Exercise 3.4:

The midpoint algorithm shall be applied to drawing a part of the graph of the function $y = -a\sqrt{x} + b$ $(a, b \in \mathbb{N}^+)$.

(a) For which x-values is the slope between -1 and 0?

(b) Rewrite the function in a suitable implicit form $F(x, y) = 0$. Use $d = F(x, y)$ as a decision variable to develop the midpoint algorithm. How does d change depending on whether the eastern (E) or the southeastern (SE) point was drawn in the previous step of the midpoint algorithm?

(c) How should the initial value d_{init} for d be chosen, if $(x_0, y_0) = (a^2, -a^2 + b)$ is the first point of the curve to be drawn?

(d) How can the rational values for the decision variable be avoided?

Solution (sketch):

(a) $y' = \frac{-a}{2\sqrt{x}} < 0$.

$y' \geq -1$ if and only if $x \geq \frac{a^2}{4}$.

(b) $y = -a\sqrt{x} + b \Leftrightarrow (y - b) = -a\sqrt{x}$. For $x > 0$ and $y \leq b$, this equivalent to $(y - b)^2 = a^2 x$, and therefore $a^2 x - (y - b)^2 = 0$.

Thus, choose

$$d = F(x, y) = a^2 x - (y - b)^2.$$

Case 1: E, i.e. $(x_{p+1}, y_{p+1}) = (x_p + 1, y_p)$ was the pixel drawn after (x_p, y_p).

$$d_{\text{new}} = F\left(x_p + 2, y_p - \frac{1}{2}\right) = a^2 \cdot (x_p + 2) - \left(y_p - \frac{1}{2} - b\right)^2$$

$$d_{\text{old}} = F\left(x_p + 1, y_p - \frac{1}{2}\right) = a^2 \cdot (x_p + 1) - \left(y_p - \frac{1}{2} - b\right)^2$$

$$\Delta_E = d_{\text{new}} - d_{\text{old}} = a^2.$$

Case 2: SE, i.e. $(x_{p+1}, y_{p+1}) = (x_p + 1, y_p - 1)$ was the pixel drawn after (x_p, y_p).

$$d_{\text{new}} = F\left(x_p + 2, y_p - \frac{3}{2}\right) = a^2 \cdot (x_p + 2) - \left(y_p - \frac{3}{2} -\right)^2$$

$$\Delta_{SE} = d_{\text{new}} - d_{\text{old}} = a^2 + 2(y_p - b) + 2$$

(c) Initialisation:

$$d_{\text{init}} = F\left(a^2 + 1, -a^2 + b - \frac{1}{2}\right) = \frac{1}{4}$$

(d) Consider the decision variable $D = 4d$ instead of d.