The Inherent Indistinguishability in Fuzzy Systems

Frank Klawonn[1] and Rudolf Kruse[2]

[1] Department of Computer Science
University of Applied Sciences Braunschweig/Wolfenbüttel
Salzdahlumer Str. 46/48
D-38302 Wolfenbuettel, Germany
f.klawonn@fh-wolfenbuettel.de

[2] Department of Computer Science
Otto-von-Guericke University
Universitätsplatz 2
D-39106 Magdeburg, Germany
kruse@iik.cs.uni-magdeburg.de

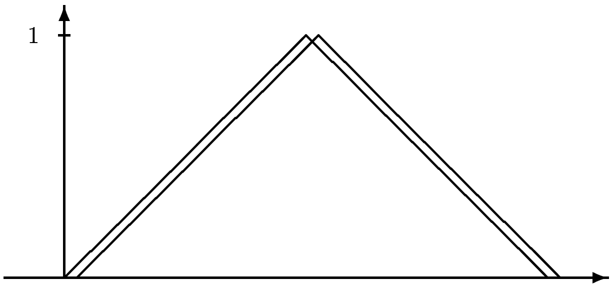
Abstract. This paper provides an overview of fuzzy systems from the viewpoint of similarity relations. Similarity relations turn out to be an appealing framework in which typical concepts and techniques applied in fuzzy systems and fuzzy control can be better understood and interpreted. They can also be used to describe the indistinguishability inherent in any fuzzy system that cannot be avoided.

1 Introduction

In his seminal paper on fuzzy sets L.A. Zadeh [14] proposed to model vague concepts like *big, small, young, near, far*, that are very common in natural languages, by fuzzy sets. The fundamental idea was to allow membership degrees to sets replacing the notion of crisp membership. So the starting point of fuzzy systems is the fuzzification of the mathematical concept $\in$ (*is element of*). Therefore, a fuzzy set can be seen as generalized indicator function of a set. Where a indicator function can assume only the two values zero (standing for: is not element of the set) and one (standing for: is element of the set), fuzzy sets allow arbitrary membership degrees between zero and one.

However, when we start to fuzzify the mathematical concept of being an element of a set, it seems obvious that we might also question the idea of crisp equality and generalize it to $[0,1]$-valued equalities, in order to reflect the concept of similarity. Figure 1 shows two fuzzy sets that are almost equal. From the extensional point of view, these fuzzy sets are definitely different. But from the intensional point of view in terms of modelling vague concepts they are almost equal.

In the following we will discuss the idea of introducing the concept of (intensional) fuzzified equality (or similarity). We will review some results that on



Two fuzzy sets that are almost equal.

Fig. 1. Two similar fuzzy sets

the one hand show that working with this kind of similarities leads to a better understanding of fuzzy systems and that these fuzzified equalities describe an inherent indistinguishability in fuzzy systems that cannot be overcome.

2 Fuzzy Logic

In classical logic the basics of the semantics part are truth functions for the logical connectives like $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \ldots$

Since classical logic deals with only two truth zero (false) and one (true), these truth functions can be defined in terms of simple tables as for instance for the logical connective $\wedge$ (AND):

$$\wedge : \{0,1\} \times \{0,1\} \longrightarrow \{0,1\}$$

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

In the context of fuzzy sets or fuzzy systems this restriction of a two-valued logic must be relaxed to $[0,1]$-valued logic. Therefore, the truth functions of the logical connectives must be extended from the set $\{0,1\}$ to the unit interval. Typical examples for generalized truth functions $* : [0,1] \times [0,1] \longrightarrow [0,1]$ for the logical AND $\wedge$ are:

$\alpha * \beta$	name
$\min\{\alpha, \beta\}$	minimum
$\max\{\alpha + \beta - 1, 0\}$	Łuksiewicz t-norm
$\alpha \cdot \beta$	product
$\begin{cases} \min\{\alpha, \beta\} & \text{if } \max\{\alpha, \beta\} = 1 \\ 0 & \text{otherwise} \end{cases}$	drastic product

The axiomatic framework of t-norms provides a more systematic approach to extending $\wedge$ to $[0, 1]$-valued logics. A **t-norm** $*$ is a commutative, associative, binary operation on [0,1] with 1 as unit that is non-decreasing in its arguments. The dual concept for the logical connective OR $\vee$ are t-conorms. A **t-conorm** $*$ is a commutative, associative, binary operation on [0,1] with 0 as unit that is non-decreasing in its arguments. A t-norm $*$ induces a t-conorm $\tilde{*}$ by $\alpha \tilde{*} \beta = 1 - ((1 - \alpha) * (1 - \beta))$ and vice versa.

In this paper, we will restrict our consideration to continuous t-norms. In this case, we can introduce the concept of residuated implication. $\rightarrow_*$ is called **residuated implication** w.r.t. $*$, if

$$\alpha * \beta \leq \gamma \;\Rightarrow\; \alpha \leq \beta \rightarrow_* \gamma$$

holds for all $\alpha, \beta, \gamma \in [0, 1]$.

A continuous t-norm $*$ has a unique residuated implication given by

$$\alpha \rightarrow_* \beta \;=\; \bigvee\{\lambda \in [0, 1] \mid \alpha * \lambda \leq \beta\}.$$

The **biimplication** w.r.t. to the (residuated) implication $\rightarrow_*$ is defined by

$$\alpha \leftrightarrow_* \beta \;=\; (\alpha \rightarrow_* \beta) \wedge (\beta \rightarrow_* \alpha).$$

The **negation** w.r.t. to the (residuated) implication $\rightarrow_*$ is defined by

$$\neg_* \alpha \;=\; \alpha \rightarrow_* 0.$$

The most common examples for t-norms and induced logical connectives are:

1. $\alpha * \beta \;=\; \min\{\alpha, \beta\}$

$$\alpha \rightarrow \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{otherwise} \end{cases}$$

$$\alpha \leftrightarrow = \begin{cases} 1 & \text{if } \alpha = \beta \\ \min\{\alpha, \beta\} & \text{otherwise} \end{cases}$$

$$\neg \alpha = \begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$$

2. $\alpha * \beta \;=\; \max\{\alpha + \beta - 1, 0\}$

$$\alpha \rightarrow \beta = \min\{1 - \alpha + \beta, 1\}$$
$$\alpha \leftrightarrow \beta = 1 - |\alpha - \beta|$$
$$\neg \alpha = 1 - \alpha$$

3. $\alpha * \beta \;=\; \alpha \cdot \beta$

$$\alpha \to \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \frac{\beta}{\alpha} & \text{otherwise} \end{cases}$$

$$\alpha \leftrightarrow \beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ \frac{\min\{\alpha,\beta\}}{\max\{\alpha,\beta\}} & \text{otherwise} \end{cases}$$

$$\neg\alpha = \begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$$

If $[\![A]\!]$ denotes the truth value of the logical formula A, then the truth functions for quantifiers are given by

$$[\![(\forall x)(A(x))]\!] \;=\; \bigwedge_x [\![A(x))]\!] \quad \text{and} \quad [\![(\exists x)(A(x))]\!] \;=\; \bigvee_x [\![A(x))]\!]$$

It should be mentioned that these concepts of $[0,1]$-valued logics lead to interesting generalizations of classical logic from the purely mathematical point of view. However, the assumption of truth-functionality, i.e. that the truth value of a complex logical formula depends only on the truth values of its compound elements, leads to certain problems. Truth-functionality implies a certain independence assumption between the logical formulae. Like in probability theory, independence is a very strong assumption that is seldom satisfied in practical applications.

Already the simple example of three-valued Łukasiewicz logic illustrates this problems. The third truth value u in this logic stands for *undetermined*. The logical connective $\wedge$ is defined canonically by the following truth function:

$$* : \{0,u,1\} \times \{0,u,1\} \longrightarrow \{0,u,1\}$$

A	B	$A * B$
0	0	0
0	u	0
0	1	0
u	0	0
u	u	u
u	1	u
1	0	0
1	u	u
1	1	1

The following simple example shows the problem caused by truth functionality.

Proposition	Meaning	$[\![\ldots]\!]$
A	The German chancellor will be in Berlin on 30 November 2010.	u
B	It will rain in Berlin on 30 November 2010.	u
$A \wedge B$	...	u
$A \wedge \neg A$	...	0

A and B are independent (hopefully). A and $\neg A$ are definitely not. So there is no consistent way of assigning a truth value to a logical conjunction of two statements based only on their truth values, since A, B and $\neg A$ all have the truth value *undetermined*, as well as the logical statement $A \wedge B$, whereas $A \wedge \neg A$ should be assigned the truth value false.

However, in applications of fuzzy systems like fuzzy control this problem usually plays only a minor role, because certain independence assumptions are satisfied there by the structure of the considered formal framework.

3 Similarity Relations

Before introducing the notion fuzzified equality or similarity, we briefly review how mathematical concepts can be fuzzified in a straight forward way.

We interpret the membership degree $\mu(x)$ of an element x to a fuzzy set μ as the truth value of the statement *x is an element of μ*.

$$\mu(x) \;=\; [\![x \in \mu]\!]$$

When we want to consider the fuzzified version of an axiom A (in classical logic), we take into account that axioms are assumed to be true, i.e. $[\![A]\!] = 1$. Also, axioms are very often of the form $B \to C$. Using residuated implications, we have

$$[\![B \to C]\!] = 1 \;\Leftrightarrow\; [\![B]\!] \leq [\![C]\!]$$

Having these facts in mind, it is obvious how to interpret an axiom in a $[0,1]$-valued logic. As a concrete example, we consider the notion of equivalence relations.

classical logic	fuzzy logic
relation: $E \subseteq X \times X$	fuzzy relation: $E : X \times X \longrightarrow [0,1]$
$(x,x) \in E$	$E(x,x) = 1$
$(x,y) \in E \Rightarrow (y,x) \in E$	$E(x,y) \leq E(y,x)$ (thus $E(x,y) = E(y,x)$)
$(x,y) \in E \wedge (y,z) \in E \Rightarrow (x,z) \in E$	$E(x,y) * E(y,z) \leq E(x,z)$

A fuzzy relation

$$E : X \times X \longrightarrow [0,1]$$

on a set X satisfying the three previously mentioned axioms is called an **similarity relation** [15, 11]. Depending on the choice of the operation $*$, sometimes E is also called an indistinguishability operator [13], fuzzy equality (relation) [2, 7], fuzzy equivalence relation [12] or proximity relation [1].

A fuzzy relation E is a similarity relation w.r.t. the Łukasiewicz t-norm, if and only if $1 - E$ is a pseudo-metric bounded by 1. A fuzzy relation E is an similarity relation w.r.t. the minimum, if and only if $1 - E$ is an ultra-pseudo-metric bounded by 1. Any (ultra-)pseudo-metric δ bounded by 1 induces an similarity relation w.r.t. the Łukasiewicz t-norm (minimum) by $E = 1 - \delta$.

Extensionality in the context of similarity relation means to respect the similarity relation: Equal (similar) elements should lead to equal (similar) results. The classical property: $x \in M \wedge x = y \quad \Rightarrow \quad y \in M$ leads to the following definition.

A fuzzy set μ is called **extensional** w.r.t. an similarity relation E, if

$$\mu(x) * E(x, y) \ \leq \ \mu(y)$$

holds.

Let $E : X \times X \to [0, 1]$ be an similarity relation on the set X. The extensional hull $\hat{\mu}$ of the fuzzy set μ is smallest extensional fuzzy set containing μ given by

$$\hat{\mu}(y) \ = \ \bigvee_{x \in X} (\mu(x) * E(x, y)).$$

$$y \in \mu \ \Leftrightarrow \ (\exists x \in X)(x \in \mu \wedge x = y)$$

The extensional hull of an ordinary set is the extensional hull of its indicator function and can be understood as the (fuzzy) set of points that are equal to at least one element in the set.

As an example consider the similarity relation $E(x, y) = 1 - \min\{|x - y|, 1\}$. The extensional hulls of a single point and an interval are shown in figure 2.

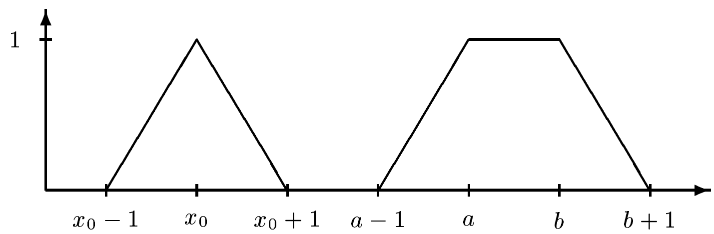


Fig. 2. The extensional hulls of the point x_0 and of the interval $[a, b]$.

Extensional hulls of points w.r.t.

$$E(x, y) = 1 - \min\{|x - y|, 1\}$$

always have a support of length two. In order to maintain the degrees of similarity (and the membership degrees), when changing the measurement unit (seconds instead of hours, miles instead of kilometers,...), we have to take a scaling into account:

$$E(x, y) = 1 - \min\{c \cdot |x - y|, 1\}$$

When we take a closer look at the concept of similarity relations, we can even introduce a more general concept of scaling. Similarity relations can be used to model indistinguishability. There are two kinds of indistinguishability, we have to deal with in typical fuzzy control applications.

Enforced indistinguishability is caused by limited precision of measurement instruments, (imprecise) indirect measurements, noisy data, ...

Intended indistinguishability means that the control expert is not interested in more precise values, since a higher precision would not really lead to an improved control.

Both kinds of indistinguishability might need a local scaling as the following example of designing an air conditioning system shows.

temperature (in °C)	scaling factor	interpretation
< 15	0.00	exact value meaningless (much too cold)
15-19	0.25	too cold, but not too far away from the desired temperature, regulation need not be too sensitive
19-23	1.50	very sensitive, near the optimal value
23-27	0.25	too warm, but not too far away from the desired temperature, regulation need not be too sensitive
> 27	0.00	exact value meaningless (much too hot)

When we apply these different scaling factors to our temperature domain, this has the following consequences, when we consider the similarity relation induced by the scaled distance. In order to determine how dissimilar two temperatures are, we do note compute their difference directly, but in the scaled domain, where the range up to 15 is shrunk to a single point, the range between 15 and 19 is shrunk by the factor 0.25, the range between 19 and 23 is stretched by the factor 1.5 and so on. The following table shows the scaled distances of some example values for the temperature.

pair of values	scal. factor	transformed distance	similarity degree
(x, y)	c	$\delta(x,y) = \|c \cdot x - c \cdot y\|$	$E(x,y) = 1 - \min\{\delta(x,y), 1\}$
(13,14)	0.00	0.000	1.000
(14,14.5)	0.00	0.000	1.000
(17,17.5)	0.25	0.125	0.875
(20,20.5)	1.50	0.750	0.250
(21,22)	1.50	1.500	0.000
(24,24.5)	0.25	0.125	0.875
(28,28.5)	0.00	0.000	1.000

Figures 3 and 4 show examples of extensional hulls of single points.

The idea of piecewise constant scaling functions can be extended to arbitrary scaling functions in the following way [4]. Given an integrable scaling function: $c : R \to [0, \infty)$. If we assume that we have for small values ε that the transformed distance between x and $x + \varepsilon$ is given by

$$\delta(x, x + \varepsilon) \approx c \cdot |\varepsilon|,$$

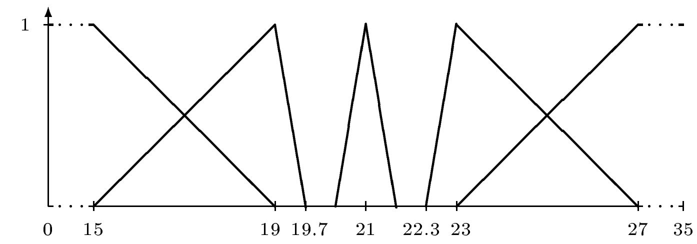


Fig. 3. The extensional hulls of the points 15, 19, 21, 23 and 27.

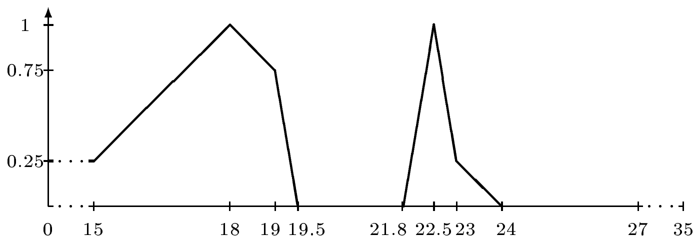


Fig. 4. The extensional hulls of the points 18.5 and 22.5.

then the transformed distance induced by the scaling function c can be computed by

$$\left| \int_x^y c(s)\, ds \right| .$$

4 The Inherent Indistinguishability in Fuzzy Systems

In this section we present some results [4, 6, 5] on the connection between fuzzy sets and similarity relations.

Given a set $\mathcal{A}$ of fuzzy sets ('a fuzzy partition'). Is there an similarity relation E s.t. all these fuzzy sets are extensional w.r.t. E? The answer to this question is positive.

$$E_{\mathcal{A}}(x, y) \;=\; \bigwedge_{\mu \in \mathcal{A}} (\mu(x) \leftrightarrow \mu(y))$$

is the coarsest similarity relation making all fuzzy sets in $\mathcal{A}$ extensional.

We go a step further and consider a given set $\mathcal{A}$ of normal fuzzy sets (that have membership degree one for at least one point). Is there an similarity relation E s.t. all these fuzzy sets can be interpreted as extensional hulls of points?

Let $\mathcal{A}$ be a set of fuzzy sets such that for each $\mu \in \mathcal{A}$ there exists $x_\mu \in X$ with $\mu(x_\mu) = 1$. There is an similarity relation E, such that for all $\mu \in \mathcal{A}$ the extensional hull of the point x_μ coincides with the fuzzy set μ, if and only if

$$\bigvee_{x \in X} (\mu(x) * \nu(x)) \;\le\; \bigwedge_{y \in X} (\mu(y) \leftrightarrow \nu(y))$$

holds for all $\mu, \nu \in \mathcal{A}$.

In this case, $E = E_{\mathcal{A}}$ is the coarsest similarity relation for which the fuzzy sets in $\mathcal{A}$ can be interpreted as extensional hulls of points.

If the fuzzy sets are pairwise disjoint ($\mu(x) * \nu(x) = 0$ for all x), then the condition of the previous theorem is always satisfied. For the Łukasiewicz t-norm this means

$$\mu(x) + \nu(x) \le 1.$$

Let $\mathcal{A}$ be a non-empty, at most countable set of fuzzy sets such that each $\mu \in \mathcal{A}$ satisfies:

- There exists $x_\mu \in R$ with $\mu(x_\mu) = 1$.
- μ (as a real-valued function) is increasing on $(-\infty, x_\mu]$.
- μ is decreasing on $[x_\mu, -\infty)$.
- μ is continuous.
- μ is differentiable almost everywhere.

There exists a scaling function $c : R \to [0, \infty)$ such that for all $\mu \in \mathcal{A}$ the extensional hull of the point x_μ w.r.t. the similarity relation

$$E(x, y) \;=\; 1 - \min\left\{ \left| \int_x^y c(s)\, ds \right|, 1 \right\}$$

coincides with the fuzzy set μ, if and only if

$$\min\{\mu(x), \nu(x)\} > 0 \;\Rightarrow\; \left| \frac{d\mu(x)}{dx} \right| = \left| \frac{d\nu(x)}{dx} \right|$$

holds for all $\mu, \nu \in \mathcal{A}$ almost everywhere. In this case,

$$c : R \to [0, \infty), \quad x \mapsto \begin{cases} \left| \frac{d\mu(x)}{dx} \right| & \text{if } \mu(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

can be chosen as the (almost everywhere well-defined) scaling function.

Figure 5 shows a typical example of a choice of fuzzy sets. For this kind of fuzzy partition a scaling function exists, such that the fuzzy sets can be represented as extensional hulls of points.

There is another explanation, why fuzzy sets are very often chosen as shown in this figure. The expert who specifies the fuzzy sets and the rules for the fuzzy system is assumed to specify as few rules as possible. When he has chosen one point (inducing a fuzzy set as its extensional hull), taking the similarity relations into account, this single point provides some information for all points that have non-zero similarity/indistinguishability to the specified point. Therefore, the next point must be specified, when the similarity degree (membership degree of the corresponding fuzzy set) has dropped to zero.

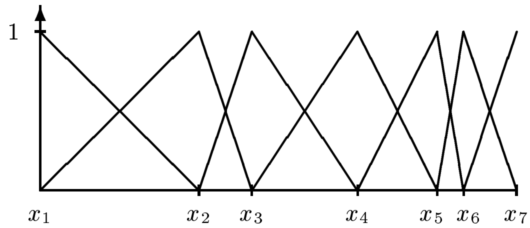


Fig. 5. Fuzzy sets for which a scaling function can be defined.

5 Similarity Relations and Fuzzy Functions

If we assume that the fuzzy sets in fuzzy control applications represent (vague) points, then each rule specifies a point on the graph of the control function. A rule is typically of the form

If x_1 is μ_1 and ... and x_n is μ_n, then y is ν.

where $x_1, \dots, x_n$ are input variables and y is the output variable and $\mu_1, \dots, \mu_n$ and ν are suitable fuzzy sets.

In this way, fuzzy control can be seen as interpolation in the presence of vague environments characterized by similarity relations. A function $f : X \longrightarrow Y$ is extensional w.r.t. to the similarity relations E and F on X and Y, respectively, if

$$E(x, x') \leq F(f(x), f(x'))$$

holds for all $x, x' \in X$.

Interpreting fuzzy control in this way, defuzzification means to find an extensional function that passes through the points specified by the rule base. It can be shown [10] that the centre of gravity defuzzification method is a reasonable heuristic technique, when the fuzzy sets and the rules are 'well-behaved'. From a theoretical point of view, we have to find a function through the given control points that is Lipschitz continuous (w.r.t. the metrics induced by the equality relations) with Lipschitz constant 1.

Since fuzzy controllers usually have multiple inputs, it is necessary to combine the similarity relations to a single similarity relation in the product space. The canonical similarity relation on a product space is given by [9]

$$E((x_1, \dots, x_p), (x'_1, \dots, x'_p)) \;=\; \min_{i \in \{1,\dots,p\}} \{E_i(x_i, x'_i)\}.$$

In terms of fuzzy control this means that for a single rule, the membership degrees of an input would be combined using the minimum.

Viewing fuzzy control in this way, the specification of (independent) fuzzy sets respectively similarity relations means that the indistinguishabilities on the different inputs are independent. Although this is an unrealistic assumption, fuzzy control works quite well. The independence problem is partly solved, by using a fine granularity everywhere and specifying more rules.

Finally, we would like to emphasize that, even if the fuzzy sets are chosen in such a way that they cannot be interpreted as extensional hulls of points, similarity relations play an important role. We can always compute the coarsest similarity relations making all fuzzy sets extensional. It can be shown under quite general assumptions [6] that

- the output of a fuzzy system does not change, when we replace the input by its extensional hull and
- the output (before defuzzification) is always extensional.

6 Conclusions

We have shown that similarity relations provide an interesting framework to better understand the concepts underlying fuzzy systems and fuzzy control. They can also be used to characterize the indistinguishability that is inherent in any fuzzy system. Exploiting the ideas of the connection between fuzzy systems and similarity relations further leads also to interesting connections to fuzzy clustering [3] and to understanding fuzzy control as knowledge-based interpolation [8] which leads to a much stricter framework of fuzzy systems in which inconsistencies can be avoided easier [5]

References

1. Dubois, D., Prade, H.: Similarity-Based Approximate Reasoning. In: Zurada, J.M., Marks II, R.J., Robinson, C.J. (eds.): Computational Intelligence Imitating Life. IEEE Press, New York (1994), 69-80
2. Höhle, U., Stout, L.N.: Foundations of Fuzzy Sets. Fuzzy Sets and Systems 40 (1991), 257-296
3. Höppner, F., Klawonn, F., Kruse, R., Runkler. T.: Fuzzy Cluster Analysis. Wiley, Chichester (1999)
4. Klawonn, F.:Fuzzy Sets and Vague Environments. Fuzzy Sets and Systems 66 (1994), 207-221
5. Klawonn, F.: Fuzzy Points, Fuzzy Relations and Fuzzy Functions. In: Novák, V., Perfilieva, I. (eds.): Discovering the World with Fuzzy Logic. Physica-Verlag, Heidelberg (2000), 431-453
6. Klawonn, F., Castro, J.L.: Similarity in Fuzzy Reasoning. Mathware and Soft Computing 2 (1995), 197-228
7. Klawonn, F., Kruse, R.:Equality Relations as a Basis for Fuzzy Control. Fuzzy Sets and Systems 54 (1993), 147-156
8. Klawonn, F., Gebhardt, J., Kruse, R.: Fuzzy Control on the Basis of Equality Relations – with an Example from Idle Speed Control. IEEE Transactions on Fuzzy Systems 3 (1995), 336-350
9. Klawonn, F., Novák, V.: The Relation between Inference and Interpolation in the Framework of Fuzzy Systems. Fuzzy Sets and Systems 81 (1996), 331-354

10. Kruse, R., Gebhardt, J., Klawonn, F.: Foundations of Fuzzy Systems. Wiley, Chichester (1994)
11. Ruspini, E.H.: On the Semantics of Fuzzy Logic. Intern. Journ. of Approximate Reasoning 5 (1991), 45-88
12. Thiele, H., Schmechel, N.: The Mutual Defineability of Fuzzy Equivalence Relations and Fuzzy Partitions. Proc. Intern. Joint Conference of the Fourth IEEE International Conference on Fuzzy Systems and the Second International Fuzzy Engineering Symposium, Yokohama (1995), 1383-1390
13. Trillas, E., Valverde, L.: An Inquiry into Indistinguishability Operators. In: Skala, H.J., Termini, S., Trillas, E. (eds.): Aspects of Vagueness. Reidel, Dordrecht (1984), 231-256
14. Zadeh, L.A: Fuzzy Sets. Information and Control 8 (1965), 338-353.
15. Zadeh, L.A.: Similarity Relations and Fuzzy Orderings. Information Sciences 3 (1971), 177-200