Equality Relations as a Basis for Fuzzy Control

F. Klawonn R. Kruse
Department of Computer Science
Technical University of Braunschweig
W-3300 Braunschweig, Germany

Abstract: The aim of this paper is to introduce a fuzzy control model with well–founded semantics in order to explain the concepts applied in fuzzy control. Assuming that the domains of the input– and output variables for the process are endowed with equality relations, that reflect the indistinguishability of values lying closely together, the use of triangular and trapezoidal membership functions can be justified and max–$\sqcap$ inference where $\sqcap$ is a t–norm turns out to be a consequence of our model. Distinguishing between a functional and a relational view of the control rules it is possible to explain when defuzzification strategies like MOM or COA are appropriate or lead to undesired results.

Keywords: Fuzzy control; equality relation

1 Introduction

The basic techniques of fuzzy control were already known in 1974 [18, 21], but due to the large number of successful applications of fuzzy controllers in recent years, especially in Japan, the interest of both practitioners and theorists in fuzzy control is growing. In opposition to classical control theory which is usually based on a mathematical model of the process the idea of fuzzy control is to simulate a human expert by translating linguistic if–then control rules into a control function. As a result of a benchmark found in the community of specialists in adaptive control this methods appeared to have positively surprising properties of robustness and can compete with advanced control methods such as supervised adaptive control [1].

In order to understand how and why fuzzy control is an appropriate control technique, it is necessary to provide a well–founded semantic background for the applied concepts, enabling us to explain what the specific fuzzy sets mean, where they come from and how we have to operate with these fuzzy sets. Some of the concepts applied in fuzzy control are based on a very intuitive understanding of fuzzy set theory without the use of a clear model which motivates and justifies these concepts. As a well known example consider the defuzzification method center of area in the classical max–min controller, which gives good results due to its good interpolation properties but which cannot be justified in a logical calculus. So there is a need to provide underlying semantics.

One way to attack this problem is to use methods of approximate reasoning. Generally, approaches to the foundations of fuzzy control interpret the control rules as inference rules leading to a conjunctive combination of the rules and, in case of a possibilistic interpretation of the fuzzy sets, resulting in the use of the Gödel–relation for the inference procedure [2, 14]. The interpretation of the rules and the fuzzy sets is a crucial point for providing a semantic background for fuzzy control. Since there are various interpretations for the rules as inference schemes [3] and for the origin of the fuzzy sets [5, 6, 13], the appropriate model has to be chosen carefully.

An alternative approach not based on inference methods applied in approximate reasoning is to embed fuzzy control in the classical interpolation and approximation theoretic approaches [4]. From this view fuzzy control helps to define the input–output function by using additional expert information such as linguistic rules, approximate input–output tuples etc. This area of 'knowledge based interpolation' is yet not developed.

In our approach we will also interpret fuzzy control rules as a (partial) specification of a control function or, more generally, of a control relation which has to be approximated or interpolated on the basis of control rules. We assume that the domains of the control variables are endowed with equality relations reflecting the fact that values that are lying closely together cannot be very well distinguished. Assuming this indistinguishability as the only source where fuzzyness comes into our model we are able to explain the meaning of triangular or trapezoidal membership functions and can elucidate the use of max–min inference or, more generally, max–$\sqcap$ inference where $\sqcap$ is a t–norm.

The paper is organized as follows. In section 2 we will give the formal definition of an equality relation, motivate and explain the use of equality relations with examples, and introduce modified definitions which respect the equality relations for concepts like singletons, subsets, functions, and relations. In section 3 the ideas of section 2 are applied to fuzzy control explaining why triangular and trapezoidal membership functions play a special role in fuzzy control not only because of their simplicity and how the max–min inference rule can been seen as a consequence of our fuzzy control model. Section 4 is a discussion about appropriate defuzzification rules showing that the application of the center–of–area or the mean–of–max method reflects a functional view of the the specified fuzzy control strategy that has to be taken into account for the design of the rules.

2 Equality relations

Equality relations [8, 10] reflect the idea that objects cannot always be well distinguished, wherefore they are also called indistinguishability operators [11, 20]. Throughout the rest of this paper let $\sqcap$ denote a t-norm (see f.e. [12]).

Definition 2.1 An *equality relation* on a set X (with respect to the t-norm $\sqcap$) is a mapping $E : X \times X \to [0,1]$ satisfying the following axioms

(i)	$E(x,x) = 1$	(total existence)
(ii)	$E(x,y) = E(y,x)$	(symmetry)
(iii)	$E(x,y) \sqcap E(y,z) \leq E(x,z)$	(transitivity).

The value $E(x,y)$ is to be understood as the degree to which x and y are equal or indistinguishable. The total existence corresponds to the statement that x is equal to x to the degree 1. Total existence is not always required [10]. In the case of $E(x,x) < 1$ the value $E(x,x)$ would be interpreted as the degree to which x exists in X or belongs to X, i.e. $E(x,x)$ reflects a membership degree. Of course, an equality relation should be symmetric. Interpreting the t-norm $\sqcap$ as a conjunction, transitivity can be read as if x is equal to y and y is equal to z then x must be equal to z.

The most simple example of an equality relation is the one induced by the crisp equality on X

$$E(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.2 Let $\alpha \sqcap \beta = \alpha * \beta = \max\{\alpha + \beta - 1, 0\}$ be the t–norm that leads to Łukasiewicz logic [7]. Let (X, δ) be a metric space. The metric δ induces an equality relation $E_\delta : X \times X \to [0,1]$ on X by

$$E_\delta(x,y) = 1 - \min\{\delta(x,y), 1\}.$$

E_δ obviously satisfies the axioms of total existence and symmetry. E_δ is also transitive since

$$\begin{aligned} E_\delta(x,y) * E_\delta(y,z) &= \max\{1 - \min\{\delta(x,y),1\} + 1 - \min\{\delta(y,z),1\} - 1, 0\} \\ &\leq \max\{1 - \min\{\delta(x,y) + \delta(y,z), 1\}, 0\} \\ &= 1 - \min\{\delta(x,y) + \delta(y,z), 1\} \\ &\leq 1 - \min\{\delta(x,z), 1\} \\ &= E_\delta(x,z). \end{aligned}$$

Although equality relations reflect the intuitive idea of indistinguishability, we still have to justify the use of the values $E(x,y)$ by a concrete interpretation of these values not only in terms of 'degree of equality' without specifying what a degree of 0.9 means. The following two examples motivate the use of equality relations in connection with the t-norms $\wedge = \min$ and $*$, respectively.

Example 2.3 Assume X is a set of objects which can only be observed through milky glasses of a thickness ranging from 0 to 1 (for this interpretation of vague data see also [14]). Let $x, y \in X$ be two objects. We observe the isolated object x through a milky glass of thickness α. After that we observe the isolated object y through a milky glass of thickness α. If we cannot decide wether we have observed different objects x and y or twice the same object x, then we say that x and y are α–indistinguishable ($\alpha \in [0,1]$). Define

$$E(x,y) = \sup\{1-\alpha \mid x \text{ and } y \text{ are } \alpha\text{–indistinguishable}\},$$

where $\sup\emptyset = 0$. Obviously, $E(x,x) = 1$ and $E(x,y) = E(y,x)$ hold. Let x and y be α–indistinguishable and let y and z be β–indistinguishable. Let $\gamma = \max\{\alpha, \beta\}$. Since $\gamma \geq \alpha$ holds, x and y are γ–indistinguishable. The same argument applies to y and z. This means that if we observe x, y, and z through a milky glass of thickness γ we can neither distinguish x from y nor y from z, which implies that we cannot distinguish x from z (except in the case when we have to deal with Poincaré's paradox [19], i.e. $A = B$, $B = C$, but $A \neq C$, which we will not consider here). In other words, x and z are γ–indistinguishable. Therefore, we obtain

$$\begin{aligned}
E(x,z) &= \sup\{1-\gamma \mid x \text{ and } z \text{ are } \gamma\text{–indistinguishable}\} \\
&\geq \sup\{1-\gamma \mid x \text{ and } y \text{ are } \gamma\text{–indistinguishable and } y \text{ and } z \text{ are } \gamma\text{–indistinguishable}\} \\
&\geq \sup\{1-\max\{\alpha,\beta\} \mid x \text{ and } y \text{ are } \alpha\text{–indistinguishable and } y \text{ and } z \text{ are } \beta\text{–indistinguishable}\} \\
&= \sup\{\min\{1-\alpha, 1-\beta\} \mid x \text{ and } y \text{ are } \alpha\text{–indistinguishable and } y \text{ and } z \text{ are } \beta\text{–indistinguishable}\} \\
&= \min\{E(x,y), E(y,z)\}
\end{aligned}$$

which means that E is an equality relation with respect to the t–norm $\wedge$.

Example 2.4 Assume X is a set of objects which are characterized by what we can see on their surfaces. The shape of the surface and the size of the surface is for all $x \in X$ the same, for example a square of length 1. (The objects in X could be sheets of paper.) We call x and y α–indistinguishable ($\alpha \in [0,1]$) if there is an area of measure α in the unit square such that if we can only observe this area of the corresponding squares of x and y we cannot distinguish these squares. Define

$$E(x,y) = \sup\{\alpha \mid x \text{ and } y \text{ are } \alpha\text{–indistinguishable}\}.$$

Obviously, $E(x,x) = 1$ and $E(x,y) = E(y,x)$ hold. Let x and y be α–indistinguishable and let y and z be β–indistinguishable, i.e. there are areas A and B of measure α and β, respectively, in the square such that we cannot distinguish x from y, y from z, respectively, if we only observe the respective areas of the corresponding squares. The intersection of the two areas A and B has at least measure $\alpha+\beta-1$. Since we cannot distinguish x from z if we can only observe the area $A \cap B$ of the corresponding squares, x and z are $(\alpha+\beta-1)$–indistinguishable. Therefore, we obtain

$$
\begin{aligned}
E(x,y) * E(y,z) &= \sup\{\max\{0, \alpha+\beta-1\} \mid && x \text{ and } y \text{ are } \alpha\text{–indistinguishable} \\
& && \text{and} \\
& && y \text{ and } z \text{ are } \beta\text{–indistinguishable}\} \\
&\leq \sup\{\gamma \mid x \text{ and } z \text{ are } \gamma\text{–indistinguishable}\} \\
&= E(x,z)
\end{aligned}
$$

which means that E is an equality relation with respect to the t–norm $*$.

Examples 2.3 and 2.4 motivate the transitivity condition for equality relations with respect to the t–norms $\wedge$ and $*$. But in fact, for fuzzy control this transitivity condition is not needed except that we will require that concepts like subsets have to be extensional, and the transitivity reflects the extensionality of equality relations.

When we consider sets endowed with equality relations we have to take into account that all concepts regarding these sets have to respect the equality relations, a property called *extensionality*. For example, if x and y are equal to some degree greater zero but we know a subset such that x belongs totally and y does absolutely not belong to this subset, then we could use this subset to distinguish x and y contradicting the assumption that x and y are equal to some degree greater zero. Therefore, if x belongs to a certain degree to a subset also y has to belong in some way to this subset.

Definition 2.5 Let E be an equality relation on X. A fuzzy set $\mu : X \to [0,1]$ is *extensional* if

$$\mu(x) \sqcap E(x,y) \leq \mu(y)$$

holds for all $x, y \in X$.

$\mu(x)$ is the membership degree of x in μ. If E is the equality relation induced by the crisp equality then any fuzzy set in X is also extensional. The extensionality condition for fuzzy sets reflects the above described idea of respecting the equality relation.

Special subsets of a set are those sets called singletons containing only one element.

Definition 2.6 Let E be an equality relation on X and let $x_0 \in X$. The *singleton* induced by x_0 is the extensional fuzzy set μ_{x_0} where

$$\mu_{x_0}(x) = E(x, x_0).$$

According to the transitivity of the equality relation μ_{x_0} is extensional. In fact, μ_{x_0} is the smallest extensional fuzzy set μ in (X,E) satisfying $\mu(x_0) = 1$. The generalization of definition 2.6 to extensional fuzzy sets induced by a subset is straightforward.

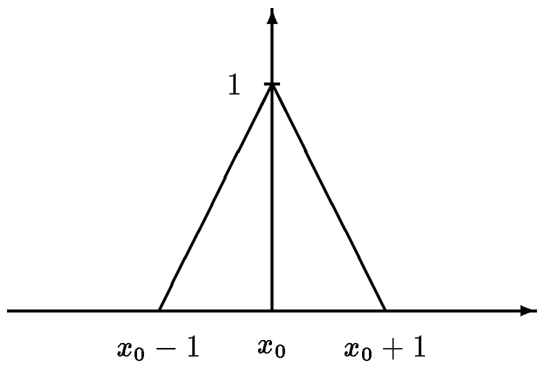

Figure 1: The singleton induced by x_0.

Definition 2.7 Let E be an equality relation on X and let $M \subseteq X$. The *fuzzy set* μ_M *induced by* M is given by

$$\mu_M(x) = \sup\{E(m, x) \mid m \in M\}.$$

μ_M is also called the *extensional hull* of M.

Again, due to the transitivity μ_M is extensional and μ_M is the smallest extensional fuzzy set containing M.

Example 2.8 Let $X = \mathbb{R}$ and let $\delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto |x - y|$ be the standard metric on $\mathbb{R}$. Consider the equality relation E_δ (with respect to the t–norm $*$), that was defined in example 2.2. The singleton μ_{x_0} induced by $x_0 \in \mathbb{R}$ is a triangular function (see figure 1), i.e.

$$\mu_{x_0}(x) = 1 - \min\{|x_0 - x|, 1\}.$$

In order to obtain triangular functions with other slopes than 1 we can consider the metric δ_c where $\delta_c(x, y) = c \cdot \delta(x, y)$, $(c > 0)$.

Let $a, b \in \mathbb{R}$. The extensional fuzzy set $\mu_{[a,b]}$ induced by the set $[a, b]$ is the trapezoidal function (see figure 2)

$$\mu_{[a,b]}(x) = \begin{cases} 1 & \text{if } a \leq x \leq b \\ \max\{1 - a + x, 0\} & \text{if } x \leq a \\ \max\{1 - x + b, 0\} & \text{if } b \leq x \end{cases}$$

Definition 2.9 Let $E_1, \ldots, E_n$ be equality relations on $X_1, \ldots, X_n$, respectively. The equality relation $E = E_1 \times \ldots \times E_n$ on the cartesian product $X_1 \times \ldots \times X_n$ is given by

$$E((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = E_1(x_1, y_1) \sqcap \ldots \sqcap E_n(x_n, y_n).$$

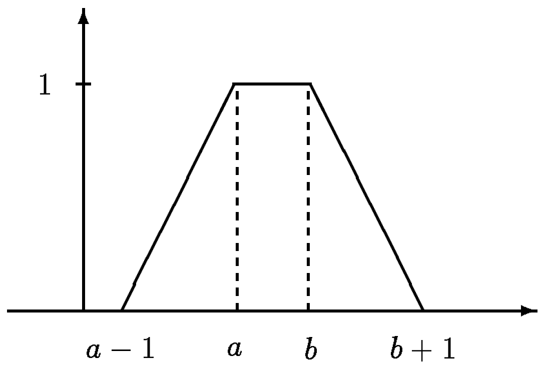

Figure 2: The fuzzy set induced by $[a, b]$.

It is easy to verify that E is an equality relation on $X_1 \times \ldots \times X_n$.

If X and Y are endowed with equality relations and R is a fuzzy relation (a fuzzy subset of $X \times Y$), then we should expect that if the pair (x, y) belongs to R to a certain degree and x' is equal to x to some degree, then also the pair (x', y) should belong to R to a certain degree. The same argument should apply when we exchange the roles of X and Y. This motivates the following definition, which requires a relation to be extensional.

Definition 2.10 Let E and F be equality relations on X and Y, respectively. A *relation* on $(X, E) \times (Y, Z)$ is a mapping (fuzzy relation) $R : X \times Y \to [0, 1]$ satisfying the extensionality axioms

(i) $R(x, y) \sqcap E(x, x') \leq R(x', y)$

(ii) $R(x, y) \sqcap F(y, y') \leq R(x, y')$.

A mapping or morphism $f : X \to Y$ (in the classical sense) is an ordinary relation $f \subseteq X \times Y$ with the additional properties (i) if $(x, y) \in f$ and $(x, y') \in f$ then $y = y'$ is implied and (ii) for each $x \in X$ there exists a $y \in Y$ such that $(x, y) \in f$. Translating these ideas to the concept of equality relations we obtain

Definition 2.11 Let E and F be equality relations on X and Y, respectively. A *morphism* from (X, E) to (Y, F) is a relation φ on $(X, E) \times (Y, Z)$ such that

(i) $\varphi(x, y) \sqcap \varphi(x, y') \leq F(y, y')$ (singleton property)

(ii) $\sup\{\varphi(x, y) \mid y \in Y\} = 1$ (total definiteness)

hold.

A morphism corresponds to a mapping from X to Y that respects the equality relation. The value $\varphi(x, y)$ can be interpreted as the degree to which φ maps x onto y.

3 Equality relations and fuzzy control

In ordinary fuzzy control the considered data are fuzzy but the environment is crisp whereas in our approach we have ordinary tuples in a fuzzy environment (expressed in terms of equality relations). This idea is well known in quantum physics, where also the problem arises that one cannot speak of two different points if the distance between these points is less than ε [17].

We consider the following control problem. We have n input variables $\xi_1, \ldots, \xi_n$ with ranges $X_1, \ldots, X_n$, respectively, and for reasons of simplicity, one output or control variable η with range Y. The control task consists of specifying appropriate output values $\eta = y$ for given (measured) crisp inputs $\xi_1 = x_1, \ldots, \xi_n = x_n$ such that our plant operates in the intended way.

We assume that the sets $X_1, \ldots, X_n$, and Y of possible values for the variables are endowed with equality relations $E_1, \ldots, E_n$, and F, respectively. If the sets $X_1, \ldots, X_n$, and Y are intervals, the equality relations reflect the fact that in general we are not interested in 100 percent exact values or are not able do distinguish between values that are very close together. For the case of $X_1, \ldots, X_n$, and Y being intervals, the equality relations could be generated by metrics δ_c as in example 2.2. Note, that there is an implicit independence assumption for the indistinguishabilities of the input variables when we consider the product space $X_1 \times \ldots \times X_n$ and endow it with the induced equality relation, i.e. we assume that the indistinguishability of any two values $x_i^{(1)}, x_i^{(2)} \in X_i$ is independent of the value of the variable $\xi_j \in X_j$ $(j \neq i)$. This is obviously not a severe restriction. But even if we assume interacting indistinguishabilities between domains X_i and X_j, we may consider the space $X_i^{(i,j)} = X_i \times X_j$ instead of the spaces X_i and X_j, where an appropriate equality relation on $X_i^{(i,j)}$ reflecting the dependencies has to be specified. The dependency of two domains with respect to indistinguishabilty has nothing to do with dependencies between variables.

According to the equality relations the (unknown) control function should be assumed to be characterized by a morphism φ from $(X_1, E_1) \times \ldots \times (X_n, E_n)$ to (Y, F). Therefore, if φ is known and the inputs $\xi_1 = x_1, \ldots, \xi_n = x_n$ are given, for each $y \in Y$ the value $\varphi((x_1, \ldots, x_n), y)$ can be computed, that means, we obtain for each $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$ a fuzzy set $\mu^{\text{output}}_{x_1,\ldots,x_n} : Y \to [0, 1]$ by

$$\mu^{\text{output}}_{x_1,\ldots,x_n}(y) = \varphi((x_1, \ldots, x_n), y).$$

Those values y with high membership degrees represent appropriate output values for the input values $\xi_1 = x_1, \ldots, \xi_n = x_n$. In order to obtain a single output value we still have to defuzzify $\mu^{\text{output}}_{x_1,\ldots,x_n}$. In this section we only consider fuzzy control without defuzzification. Defuzzification strategies will be discussed in section 4.

Although it is in general impossible to specify φ for all pairs $((x_1, \ldots, x_n), y)$, we or the control expert might know that for certain input values $(x_1^{(i)}, \ldots, x_n^{(i)})$ the correct output values are $y^{(i)}$ (for $i = 1, \ldots, k$). We therefore assume that

$$\varphi((x_1^{(i)}, \ldots, x_n^{(i)}), y^{(i)}) = 1 \tag{1}$$

for $i = 1, \ldots, k$.

The specification of these k input–output pairs corresponds to the k control rules

If ξ_1 is (approximately) $x_1^{(i)}$ and … and ξ_n is (approximately) $x_n^{(i)}$ then η is (approximately) $y^{(i)}$. $\quad (i = 1, \ldots, k)$ (2)

The values $x_1^{(i)}, \ldots, x_n^{(i)}, y^{(i)}$ $(i = 1, \ldots, k)$ should be interpreted as singletons, i.e. to each of these values we associate an extensional fuzzy set $\mu_{x_1^{(i)}}, \ldots, \mu_{x_n^{(i)}}, \mu_{y^{(i)}}$, respectively. If the equality relations are induced by metrics as in example 2.2, these extensional fuzzy sets are represented by triangular functions.

The problem we have to solve is the following. Given arbitrary input values $\xi_1 = x_1, \ldots, \xi_n = x_n$ how can $\mu^{\text{output}}_{x_1,\ldots,x_n}$ be computed when only the above mentioned k control rules are known? According to the extensionality of φ a lower bound can be computed for $\mu^{\text{output}}_{x_1,\ldots,x_n}$.

Theorem 3.1 *Given the above described assumptions then*

$$\mu^{\text{output}}_{x_1,\ldots,x_n}(y) \geq \max_{i=1,\ldots,k} \{\mu_{x_1^{(i)}}(x_1) \sqcap \ldots \sqcap \mu_{x_n^{(i)}}(x_n) \sqcap \mu_{y^{(i)}}(y)\}. \tag{3}$$

holds for all $y \in Y$.

Proof. Due to the extensionality of φ we obtain

$$\begin{aligned}
\mu^{\text{output}}_{x_1,\ldots,x_n}(y) &= \varphi((x_1, \ldots, x_n), y) \\
&\geq \varphi((x_1, \ldots, x_n), y^{(i)}) \sqcap F(y, y^{(i)}) \\
&\geq \varphi((x_1^{(i)}, \ldots, x_n^{(i)}), y^{(i)}) \sqcap E((x_1^{(i)}, \ldots, x_n^{(i)}), (x_1, \ldots, x_n)) \sqcap F(y, y^{(i)}) \\
&\overset{(1)}{=} E((x_1^{(i)}, \ldots, x_n^{(i)}), (x_1, \ldots, x_n)) \sqcap F(y, y^{(i)}) \\
&= E_1(x_1^{(i)}, x_1) \sqcap \ldots \sqcap E_n(x_n^{(i)}, x_n) \sqcap F(y, y^{(i)}) \\
&= \mu_{x_1^{(i)}}(x_1) \sqcap \ldots \sqcap \mu_{x_n^{(i)}}(x_n) \sqcap \mu_{y^{(i)}}(y).
\end{aligned} \tag{4}$$

for all $i = 1, \ldots, k$. Therefore, (4) implies (3). □

If $\sqcap = \wedge$ then (3) is exactly the output fuzzy set (before defuzzification) of the max–min controller. Thus we can explain the max–min– or more generally, the max–$\sqcap$ controller on the basis of equality relations.

Remark. Although we assumed that φ is a morphism we only needed the extensionality of φ for the derivation of (3). Thus it is sufficent to require that φ is a relation on $\big(((X_1, E_1) \times \ldots \times (X_n, E_n)) \times (Y, F)\big)$. If φ is assumed to be a morphism, the singleton property and the total definiteness enforce consistency conditions on the equality relations or equivalently on the fuzzy sets that represent the singletons specified in the control rules.

Considering a morphism φ and specifying this morphism partially in the form of (1) leads (in the case of equality relation generated by metrics as in example 2.2) to triangular fuzzy sets representing singletons. In the following we show how trapezoidal membership functions can be motivated when $R = \varphi$ is assumed to be a relation. Then, instead of specifying k control rules as in (2) the control rules have the following form.

$$\begin{array}{l} \textit{If } \xi_1 \in X_1^{(i)}, \ldots, \xi_n \in X_n^{(i)} \textit{ then} \\ \textit{any of the values } \eta \in Y^{(i)} \textit{ is an appropriate output value,} \end{array} \tag{5}$$

where $X_1^{(i)} \subseteq X_1, \ldots, X_n^{(i)} \subseteq X_n, Y^{(i)} \subseteq Y$, respectively, for $i = 1, \ldots, k$. In the same way as (1) corresponds to (2), (5) can be replaced by

$$R((x_1, \ldots, x_n), y) = 1 \tag{6}$$

if there exists $i \in \{1, \ldots, k\}$ such that $x_1 \in X_1^{(i)}, \ldots, x_n \in X_n^{(i)}$, and $y \in Y^{(i)}$. In the same way (3) was derived by exploiting the extensionality of φ, we now obtain from the extensionality of R

Theorem 3.2 *Given control rules in the form of (5) and assuming (6) for the relation R describimg the control task,*

$$\mu_{x_1,\ldots,x_n}^{\text{output}}(y) \geq \max_{i=1,\ldots,k} \{\mu_{X_1^{(i)}}(x_1) \sqcap \ldots \sqcap \mu_{X_n^{(i)}}(x_n) \sqcap \mu_{Y^{(i)}}(y)\}. \tag{7}$$

holds for all $y \in Y$.

In (7) the fuzzy sets $\mu_{X_1^{(i)}}, \ldots, \mu_{X_n^{(i)}}$, and $\mu_{Y^{(i)}}$ are the extensional hulls generated by the subsets $X_1^{(i)}, \ldots, X_n^{(i)}$, and $Y^{(i)}$, respectively. If these subsets are intervals and the equality relations are induced by metrics as in example 2.2 the corresponding fuzzy sets are now trapezoidal instead of triangular functions.

4 Defuzzification

In the last section we demonstrated how a fuzzy controller (without defuzzification strategy) can be described in terms of equality relations where fuzzy sets represent extensional hulls of crisp points or sets. The control strategy can be specified by a morphism or a relation, both leading to the same formula for the output fuzzy set. But when choosing a defuzzification strategy one has to take into account if the fuzzy controller was designed with a functional or a relational view in mind. For the functional view it is assumed that for each input–tuple there is a unique output value whereas in the relational view for some input tuples there can be a set of output values from which one may be chosen arbitrarily.

The aim of a defuzzification strategy is to compute a single output value from the output fuzzy set $\mu_{x_1,\ldots,x_n}^{\text{output}}$. The commonly used defuzzification methods are the mean

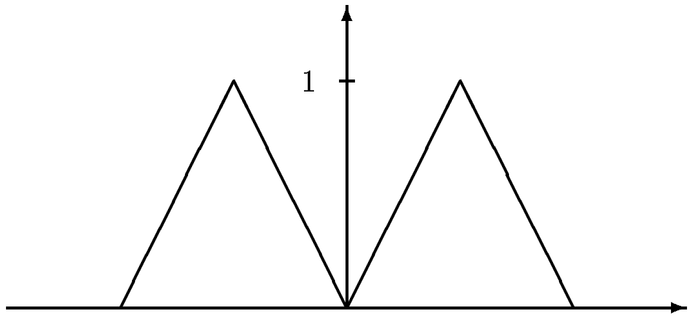

Figure 3: A fuzzy set where defuzzification strategies like MOM and COA could lead to an undesired result.

of maximum method (MOM) and the center of area method (COA) (see f.e. [16]). Both strategies are based on the implicit assumption that the control rules aim to model a morphism (function) but not a relation. In order to elucidate this, consider the following example. A set of control rules was specified for a car that should avoid collisions automatically. When the car is driving straight forward in the direction of an obstacle, the resulting output fuzzy set might look like the one in figure 3 indicating that the car should either turn left or right. But both defuzzification strategies MOM and COA lead to the undesired result that the car bumps straight into the obstacle. These methods lead to results which can be judged as correct intuitively when the output fuzzy set is unimodal, i.e. it is an increasing function until a certain point and after that it is decreasing. Thus the fuzzy set can be interpreted as representing a singleton or an interval (in terms of equality relations). But it cannot be expected to obtain such a fuzzy set if the control strategy itself in its nature represents a relation, not intended to give a unique output value for a fixed input, but to characterize a set of output values from which one can be chosen arbitrarily. In this case before applying a strategy like MOM or COA, an appropriate fuzzy subset of the output fuzzy set $\mu^{\text{output}}_{x_1,\ldots,x_n}$ has to be chosen which can be interpreted as generated by a singleton.

Therefore, when specifying the rules for the fuzzy controller, it should be made clear if these rules intend to represent a functional or a relational view, in order to be able to choose an appropriate defuzzification strategy.

5 Conclusions

We do not claim that our approach is the only way to view fuzzy control since there are other approaches based on a logical framework, for example as in [15]. But with our model the intuitive concepts applied in fuzzy control can be explained merely in terms of equality relations. For the domain of the input– and output variables it is assumed that values lying closely together are difficult to distinguish.

This indistinguishability is expressed by equality relations, which can be thought as induced by metrics when $*$ is used as the corresponding t–norm. The control rules are either given in the form of crisp input–output pairs specifying a control function partially, or as pairs of sets of input values and sets of output values reflecting the idea of a control relation. The crisp points and sets are extended to fuzzy sets in order to be extensional with respect to the equality relations. In the case of equality relations induced by metrics as in example 2.2 and 2.8 crisp points lead to triangular membership functions whereas intervals lead to trapezoidal membership functions.

The output fuzzy set computed by the max–min– or, more generally, the max–$\sqcap$–inference can be interpreted as a lower bound for the membership function induced by the actual input values and the control rules that specify a control function or relation partially. It turns out that it is important for the choice of an appropriate defuzzification strategy to determine whether the control rules are intended to represent a control function or a control relation. The commonly used defuzzification methods like MOM and COA are designed for a functional view of the control rules.

The advantage of the presented method is its sound mathematical background. Based on this model one can explain existing fuzzy control techniques. In a forthcoming paper [9] we will analyze practical aspects of fuzzy control like the derivation of the equality relation.

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