Similarity Based Reasoning

Frank Klawonn
Department of Computer Science
University of Braunschweig
38106 Braunschweig, Germany
fax (+49)(531)391-5936 E-mail klawonn@ibr.cs.tu-bs.de

Abstract

This paper is devoted to the duality between fuzzy sets and equality relations. It comprises various results that allow to interchange from one framework to the other. Finally it is shown that in fuzzy reasoning the inherent similarity characterized by equality relations cannot be avoided.

Keywords. equality relation, similarity, fuzzy reasoning

1 Introduction

Modeling human reasoning usually requires to take into account imperfect knowledge in the form of uncertainty, imprecision, vagueness, incompleteness, and partial contradictions. A great number of formal models mostly restricting to one of these phenomena is available, but none of these approaches can cover all aspects.

Fuzzy systems provide a formal framework to handle certain properties of imperfect knowledge. However, a great variety of interpretations for fuzzy sets exist, for instance in the sense of possibility distributions as a model for uncertainty or in terms of vague predicates as a model for imprecision and vagueness. The interpretation of fuzzy sets strongly affects the admissible operations on fuzzy sets. In many practical applications this fact is neglected, membership degrees are interpreted on an intuitive basis and the choice of operations is a matter of heuristics.

In this paper I concentrate on a particular interpretation of fuzzy sets where I use (fuzzy) similarity as a fundamental notion. I will clarify the duality between fuzzy sets and similarity relations. One of the consequences of this duality is that even if similarity is not the intended interpretation of fuzzy sets, one cannot avoid the effects of the similarity inherent in the fuzzy sets in fuzzy reasoning.

2 Fuzzy Sets and Similarity

Fuzzy set theory is based on a generalization or 'fuzzification' of the notion of being element of a set, i.e. for the relation $\in$ not only the (truth) values 0 and 1 are admitted, but also all intermediate values between 0 and 1. In this sense the relation $\in$ plays a central role in fuzzy set theory. Another important concept is that of equality and it is also possible to build up a framework for fuzzy sets based on a fuzzification of the notion of equality.

In order to see, how equality could be fuzzified in a suitable way, let $[\![x \approx y]\!]$ denote the 'truth' value or degree to which x and y can be considered to be equal. Thus, $\approx$ can be characterized by a mapping $E : X \times X \to [0,1]$, i.e. a binary fuzzy relation on the universe of discourse X. Of course, certain restrictions have to be assumed to accept that E reflects a fuzzification of the notion of equality. The following axioms are fundamental for the notion of equality (as well as for equivalence).

(i)	$x \approx x$	(reflexivity)
(ii)	$x \approx y \leftrightarrow y \approx x$	(symmetry)
(iii)	$x \approx y \wedge y \approx z \to x \approx z$	(transitivity)

These axioms can be translated to requirements for the fuzzy relation E representing $\approx$.

$$\begin{aligned} &(\text{E1}) \quad E_{\approx}(x,x) = 1 \\ &(\text{E2}) \quad E_{\approx}(x,y) = E_{\approx}(y,x) \\ &(\text{E3}) \quad E_{\approx}(x,y) * E_{\approx}(y,z) \le E_{\approx}(x,z) \end{aligned}$$

where $*$ is a binary operation on the unit interval with some additional properties. For reasons of simplicity we assume for the rest of this paper that $*$ is a t–norm. This motivates the following definition.

Definition 2.1 *An equality relation on a set X is a mapping $E : X \times X \to [0,1]$ fulfilling the axioms* (E1), (E2), *and* (E3).

Note that sometimes depending on the choice of the operation $*$, E is also called a similarity relation [12], indistinguishability operator [10] or proximity relation [2]. In principal, an equality relation is a special type of a fuzzy set. The choice of the t–norm $*$ depends on the interpretation of the equality relation. Consider for example a set X of photos of similar scenes, all of one inch hight. When we compare two photos, we do this by revealing them starting from the bottom. We define the degree of equality of two photos as the maximal hight (in inch) to which the photos can be revealed without showing any differences. It is easy to check that we obtain an equality relation with respect to the t–norm minimum. Even more important than the minimum–transitivity is the transitivity with respect to the Łukasiewicz t–norm $\alpha * \beta = \max\{\alpha+\beta-1, 0\}$. An equality relation E with this transitivity property induces a pseudo–metric bounded by 1 by $\delta_E(x,y) = 1-E(x,y)$, and vice verse, any pseudo–metric δ bounded by 1 defines an equality relation by $E_\delta(x,y) = 1-\delta(x,y)$. Therefore, pseudo–metrics bounded by 1 and equality relations that are transitive with respect to the Łukasiewicz t–norm are dual concepts. Note that the minimum transitivity corresponds to ultra–metrics in the same way. A canonical equality relation on the real numbers is the one induced by the standard metric, i.e. $E(x,y) = 1-\min\{|x-y|, 1\}$. An important concept for deriving other suitable equality relations are scaling factors which assign to each real number x a scaling factor $c(x)$ which gives the strength of the similarity in the neighbourhood of x. One obtains the equality relation induced by the scaled metric by

$$E(x,y) \;=\; 1 - \min\left\{ \left| \int_x^y c(s)ds \right|, 1 \right\}. \tag{1}$$

For details see [5].

The rest of this section is devoted to the connection between equality relations and fuzzy sets. A very fundamental notion for these investigations is the concept of extensionality, which stands for compatibility with an equality relation. It is motivated by the trivial observation for a classical set M: $x \in M \wedge x = y \;\to\; y \in M$. This axiom is generalized for equality relations and fuzzy sets by the following definition.

Definition 2.2 *Let E be an equality relation on X with respect to the t–norm $*$. A fuzzy set $\mu : X \to [0,1]$ is extensional if*

$$\mu(x) * E(x,y) \;\le\; \mu(y)$$

holds for all $x, y \in X$.

Before we can formulate a result about extensionality, we have to recall the notions of residuation and biresiduation. Let $*$ be a lower semi–continuous t–norm. Then the residuum and the biresiduum of $*$ are defined by

$$\to_*: [0,1]^2 \to [0,1], \qquad \alpha \to_* \beta = \sup\{\gamma \in [0,1] \mid \alpha * \gamma \le \beta\}$$

and

$$\leftrightarrow_*: [0,1]^2 \to [0,1], \qquad \alpha \leftrightarrow_* \beta = \max\{\alpha,\beta\} \to_* \min\{\alpha,\beta\},$$

respectively. If $*$ is understood as a valuation function for a conjunction, then the residuum and the biresiduum represent the valuation functions of the corresponding implication and biimplication, respectively. As a consequence of Valverde's representation theorem [11], we obtain the following result.

Theorem 2.3 *Let $(\mu_i)_{i \in I}$ be a family of fuzzy sets on the set X. Then the equality relation*

$$E(x,y) \;=\; \inf_{i \in I}\{\mu_i(x) \leftrightarrow_* \mu_i(y)\} \tag{2}$$

is greatest equality relation (satisfying the $*$–transitivity) such that the fuzzy sets μ_i are extensional for all $i \in I$.

If a fuzzy set μ is not extensional with respect to an equality relation, we can compute its extensional hull $\widehat{\mu} : X \to [0,1],\ x \mapsto \sup_{y\in X}\{\mu(y) * E(x,y)\}$, the smallest extensional fuzzy set containing μ. It is interesting to remark that a crisp set induces a fuzzy set in the form of the extensional hull of its characteristic function. If we consider the real numbers with the equality relation induced by the canonical metric we obtain a trapezoidal membership function as the extensional hull of an interval and a triangular fuzzy set as the extensional hull of a single real value [7]. We denote the extensional hull of the set M by $\mu_M(x) = \sup\{E(x,m) \mid m \in M\}$ and if M contains only one element, say $M = \{x_0\}$, by $\mu_{x_0}(x) = E(x,x_0)$.

We now turn to the question, when a family of fuzzy sets can be interpreted as the extensional hulls of one–element sets. This question was first answered in [3]. A full proof can be found in [8].

Theorem 2.4 *Let $*$ be a lower semi–continuous t–norm. Let $(\mu_i)_{i\in I}$ be a non–empty family of fuzzy sets on X and let $(x_i)_{i\in I}$ be a family of elements of X such that $\mu_i(x_i) = 1$ holds for all $i \in I$. The following two statements are equivalent.*

(i) There exists an equality relation (with respect to $*$) on X such that $\mu_i = \mu_{x_i}$ holds for all $i \in I$.

(ii) For all $i,j \in I$ the inequality

$$\sup_{x\in X}\{\mu_i(x) * \mu_j(x)\} \;\leq\; \inf_{y\in X}\{\mu_i(y) \leftrightarrow_* \mu_j(y)\} \tag{3}$$

is satisfied.

Inequality (3) simply requires that the degree of non–disjointness for any pair of the considered fuzzy sets must not exceed their degree of equality – a typical condition required for partitions. Instead of the sufficient and necessary condition (3) we can also require that the fuzzy sets are pairwise disjoint with respect to the t–norm $*$, which is a sufficient, but not necessary condition for (i). In case that (3) is satisfied one can also construct an equality relation for which (i) holds. The greatest solution is again the equality relation defined in equation (2). The smallest solution is the equality relation

$$E(x,y) \;=\; \begin{cases} 1 & \text{if } x = y \\ \sup_{i\in I}\{\mu_i(x) * \mu_i(y)\} & \text{otherwise.} \end{cases} \tag{4}$$

We observe that in the case $* = \min$ the smallest equality relation (4) and the greatest one (3) coincide for all $x \neq y$ for which

$$(\exists i \in I)(\mu_i(x) \neq \mu_i(y)) \tag{5}$$

holds. (5) simply requires that the membership degree of x and y must differ for at least one of the considered fuzzy sets. This means that the equality relation is (nearly) always unique in the case $* = \min$.

If the family of fuzzy sets is defined on a subset of the real numbers and the equality relation is assumed to be transitive with respect to the Lukasiewicz t–norm and is assumed to be defined by a scaling function like in equation (1), the following theorem holds [5].

Theorem 2.5 *Let $(\mu_i)_{i\in I}$ be an at most countable family of fuzzy sets on $\mathbb{R}$ and let $(x_0^{(i)})_{i\in I}$ be a family of real numbers such that the conditions*

(C1) $\mu_i(x_0^{(i)}) = 1$,

(C2) μ_i is a non–decreasing function on $]-\infty, x_0^{(i)}]$,

(C3) μ_i is a non–increasing function on $[x_0^{(i)}, \infty[$,

(C4) μ_i is continuous,

(C5) μ_i is almost everywhere differentiable.

are satisfied for all $i \in I$. There exists a scaling function $c : \mathbb{R} \to [0,\infty[$ such that μ_i coincides with the fuzzy set $\mu_{x_0^{(i)}}$ (for each $i \in I$) with respect to the equality relation (1), if and only if

$$\min\{\mu_i(x), \mu_j(x)\} > 0 \quad \Rightarrow \quad |\mu_i'(x)| = |\mu_j'(x)| \tag{6}$$

holds almost everywhere for all $i,j \in I$.

Note that (6) is satisfied if the fuzzy sets are chosen in such a way that $\text{card}(\{i \in I \mid \mu_i(x) > 0\}) \leq 2$ and $\sum_{i \in I} \mu_i(x) = 1$ for all x, which is very often satisfied in applications.

A related result was proved by Jacas and Recasens [4].

Theorem 2.6 *Let E be an equality relation on $\mathbb{R}$ which is transitive with respect to the Łukasiewicz t–norm. The fuzzy set μ_x is convex for all $x \in \mathbb{R}$ if and only if there exists a set J of monotonous transformations $t_j : \mathbb{R} \to \mathbb{R}$, $(j \in J)$ such that $E = \inf_{j \in J} E_j$ where the equality relation E_j is defined by $E_j(x, y) = 1 - \min\{|t_j(x) - t_j(y)|, 1\}$.*

Note that an equality relation like (1) that is induced by a scaling function c is a special type of such equality relation E_j, since it is induced by the transformation $t(x) = \int_0^x c(s)ds$.

Thiele and Schmechel [9] gave the following characterization of fuzzy partitions in the sense that they do not consider arbitrary families of fuzzy sets as in theorem 2.4, but only full fuzzy partitions, i.e. they require that for each x there is a fuzzy set such that $\mu(x) = 1$.

Theorem 2.7 *Let $*$ be an arbitrary t–norm. Let $(\mu^{(x_0)})_{x_0 \in X}$ be a family of fuzzy sets on X such that $\mu^{(x_0)}(x_0) = 1$ holds for all $x_0 \in X$. The following two statements are equivalent.*

(i) There exists an equality relation (with respect to $*$) on X such that $\mu^{(x_0)} = \mu_{x_0}$ holds for all $x_0 \in X$.

(ii) $\mu^{(x_0)}(x) = 1 \Rightarrow \mu^{(y_0)}(x) * \mu^{(y_0)}(y) \leq \mu^{(x_0)}(y)$ holds for all $x_0, y_0, x, y \in X$.

It is easy to prove that condition (ii) of theorem 2.4 implies (ii) of theorem 2.7, but not vice versa. Therefore, (ii) of theorem 2.7 is a less restrictive condition. However, theorem 2.7 does only deal with 'full' fuzzy partitions.

All the above mentioned results assume that the considered fuzzy sets are extensional hulls of single elements. Unfortunately, until now there are only very few results extending to the case that the fuzzy sets are extensional hulls of arbitrary sets.

Theorem 2.8 *Let $*$ be a lower semi–continuous t–norm. Let $(\mu_i)_{i \in I}$ be a non–empty family of fuzzy sets on X and let $X_i = \{x \in X \mid \mu_i(x) = 1\}$ be non–empty for all $i \in I$. The following three statements are equivalent.*

(i) There exists an equality relation (with respect to $*$) on X such that $\mu_i = \mu_{X_i}$ holds for all $i \in I$.

(ii) $\mu_i = \mu_{X_i}$ holds for all $i \in I$ with respect to the equality relation (2).

(iii) $\mu_i \leq \mu_{X_i}$ holds for all $i \in I$ with respect to the equality relation (2).

Although there are a lot of results that elucidate and support the interpretation of fuzzy sets on the basis of equality relations, critics might still not be willing to accept this interpretation. However, J.L. Castro [1] pointed that in typical fuzzy reasoning one might replace the input (fuzzy set) by its extensional hull without changing the conclusion. Technically speaking, we have the following result [6]. Consider a fuzzy rule of the form *if A is μ then B is ν*, where μ and ν are fuzzy sets on the universes X and Y, respectively. Assume that this rule is represented by a fuzzy relation ρ on $X \times Y$. Let us assume that $\rho = \rho_\odot$ where $\rho_\odot(x, y) = \mu(x) \odot \nu(y)$ and $\odot \in \{\min, *, \to\}$, where $*$ is a lower semi–continuous t–norm and $\to$ the corresponding residuation. Let us furthermore assume that the fuzzy rule is applied to an input fuzzy set μ' (i.e. we know A is μ') by using the formula $\nu'(y) = \sup_{x \in X}(\mu'(x) \sqcap \rho(x, y))$ (i.e. we obtain B is ν') where $\sqcap \in \{\min, *\}$.

Theorem 2.9 *Let E be a $*$–transitive equality relation on X such that the fuzzy set μ is extensional with respect to E. Let μ' be a fuzzy set on X. Then for the cases $\odot = \to$ and $\sqcap = *$, $\odot = *$ and $\sqcap = *$, and $\odot = \min$ and $\sqcap = *$ the resulting fuzzy set ν' induced by the rule if A is μ then B is ν does not change when μ' is replaced by its extensional hull.*

The above theorem states that a very precise specification of an 'input' fuzzy set or value does only make sense up to the indistinguishability characterized by the greatest equality relation for which the 'premise' fuzzy set in the rule is extensional. For chaining of rules this result does not have severe consequences. One might suspect that after each step one is allowed to build the extensional before applying the next inference and therefore the resulting fuzzy sets become wider and wider. However, the following theorem states that the resulting fuzzy set is already extensional.

Theorem 2.10 *Let F be a $*$–transitive equality relation on Y such that the fuzzy set ν is extensional with respect to F. Let μ' be a fuzzy set on X. Then for the cases $\odot = \rightarrow$ and $\sqcap = *$, $\odot = *$ and $\sqcap = *$, and $\odot = \min$ and $\sqcap = *$ the resulting fuzzy set ν' induced by the rule if A is μ then B is ν is extensional with respect to F.*

3 Conclusions and Perspectives

We have elucidated the close relations between fuzzy sets and equality relations which are suitable for the representation of similarity and indistinguishability. There are a lot of results that allow to go from fuzzy sets to corresponding equality relations and back. Theorem 2.9 shows that even if no similarity was intended when using fuzzy sets in approximate reasoning one cannot avoid the indistinguishability inherent in the fuzzy sets.

In this paper we have not treated the reasoning mechanisms in detail which are purely based on the idea of similarity. Let us remark that certain problems have to be solved, since fundamental operations cannot always be carried out in a canonical or simple way in the context of indistinguishability. For instance, although the extensional hull of the union of two (crisp) sets coincides with the extensional hull of the union of the extensional hulls, this result does not apply to the intersection of sets. A possibility to overcome this problem is the concept of local existence which allows to interpret also non–normalized fuzzy sets as extensional hulls of (locally existing) elements.

Another important topic is the derivation of fuzzy rules and therefore the corresponding equality relations from data. Fuzzy clustering techniques offer an interesting possibility, since a fuzzy cluster might be understood as the extensional hull of its prototype. Clustering algorithms like the Gustafson–Kessel and the Gath-and-Geva algorithms even compute a transformation of the canonical metric and thus are closely related to scaling functions which can be used for defining equality relations as in equation (1).

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