Fuzzy Control on the Basis of Equality Relations – with an Example from Idle Speed Control

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Abstract

The way engineers use fuzzy control in real world applications is often not coherent with an understanding of the control rules as logical statements or implications. In most cases fuzzy control can be seen as an interpolation of a partially specified control function in a vague environment, which reflects the indistinguishability of measurements or control values.

In this paper we show that equality relations turn out to be the natural way to represent such vague environments and we develop suitable interpolation methods to obtain a control function.

As a special case of our approach we obtain Mamdani's model and can justify the inference mechanism in this model and the use of triangular membership functions not only for the reason of simplified computations, and we can explain why typical fuzzy partitions are preferred. We also obtain a criterion for reasonable defuzzification strategies.

The fuzzy control methodology introduced in this paper has been applied successfully in a case study of engine idle speed control for the Volkswagen Golf GTI.

Keywords: Fuzzy control; equality relation; idle speed control

1 Introduction

The idea of fuzzy control is to model an expert or engineer who is able to control the process instead of relying on a mathematical formalization of the process itself as in classical control theory. In order to be able to build a knowledge–based model of the control actions of an engineer an appropriate mathematical framework is needed. Since human knowledge often involves imprecise and vague descriptions, appropriate approximate reasoning mechanisms have to be integrated into the knowledge–based model.

Although the concepts of fuzzy logic and especially those of fuzzy control seem to be reasonable from an intuitive point of view, criticism against these concept arises at the point where exact numbers in the form of membership degrees or the choices of appropriate operators like t–norms, t–conorms, and implication operators enter the model. Of course, it is reasonable to reject the idea of representing a concept like *old* for a certain age like 55 years in terms of the two truth values true and false. But this does not lead us to a unique and natural representation of the concept *old* as a fuzzy set with fixed membership degrees.

To overcome this problem we have to provide an interpretation of fuzzy sets for the area of fuzzy control. Such a model with well–founded semantics is a key–issue for the acceptance of fuzzy control methods for those who argue against a fuzzy system due to its heuristic background.

Although there is wide variety of approximate reasoning schemes [2, 15], only a few attempts were made to perceive fuzzy control in this light, for example as fuzzy interpolation [3]. Some misunderstandings and misinterpretations are caused by the name fuzzy *logic* controller and the idea of generalized modus ponens for the inference scheme [17], which suggest to see fuzzy control rules as logical implications [1], which would lead to other methods in fuzzy control [10].

The aim of this paper is to consider three topics.

(1) The way engineers use fuzzy control in real world applications can often be seen as an interpolation of a partially specified control function in a vague environment. We propose a model that follows this intuition.

(2) The proposed model has been applied successfully to the problem of engine idle speed control. The method provides deeper insight about how to design, tune, and evaluate fuzzy controllers.

(3) Mamdani's fuzzy control model [18] can be reformulated by using our method. With this semantics several heuristic ideas used in Mamdani's approach can be justified.

In Section 2 we define the notion of vague environments modelled by equality relations. The notion of equality relations also known under the names similarity relations or indistinguishability operators is also studied in [22, 20, 8]. We do not introduce equality relations as an abstract concept, but motivate them from an engineering point of view as a method of representing indistinguishability or similarity of points with a short distant in between. By this approach fuzzy sets can be seen as representatives of crisp values in a vague environment described by equality relations. On the other hand we show that a fuzzy partition induces a vague environment in which the corresponding fuzzy sets appear as representatives of crisp points. These ideas provide a justification of the use of triangular and trapezoidal membership functions as well as for the condition for fuzzy partitions that 'neighbouring' membership functions should overlap at membership degree 0.5.

The application of these results to fuzzy control is described in Section 3. As one consequence we derive the method proposed by Mamdani. We also obtain constraints for the control mapping induced by a fuzzy controller which can be used to evaluate defuzzification strategies. More general mathematical foundations are described in [12, 13, 14].

In Section 4 we review the method of fuzzy control based on vague environments as a systematic step–by–step approach freed from mathematical formalisms. This leads to a design and valuation method for fuzzy controllers.

The case study on idle speed control presented in Section 5 shows how these ideas can be applied to a control problem.

2 Vague Environments

This section describes the idea of vague environments and how to take them into account, when operating on them. Vague environments provide a suitable framework for analogical reasoning [6].

2.1 Equality Relations as Vague Environments

In physics and engineering sciences one often encounters the problem to handle inexact measurements. The inexactness might be caused by a limited precision of the measurement instruments or by general restrictions enforced by the experiment or process. This inexactness can be classified as indistinguishability. On the other hand, in many cases exaggerated exactness does not make sense as for example in the case of measuring the room temperature. This intentional inexactness might be called similarity. It is mainly this kind of inexactness that is inherent in fuzzy control models. However, in the following we will not distinguish between these two concept of inexactness and will call both of them indistinguishability. We consider a set of values for which we take indistinguishability into account as a vague environment.

A very common approach to cope with a vague environment $X \subseteq \mathbb{R}$ (for reasons of simplicity let us assume $X = \mathbb{R}$) is to choose a (small) number $\varepsilon > 0$ as an error or tolerance bound and to identify two values $x, x' \in X$ if their distance is at most ε, i.e. $|x - x'| \leq \varepsilon$. In that case x and x' might be called ε–indistinguishable. We can represent this ε–indistinguishability by a relation $R_\varepsilon \subseteq X \times X$ containing all pairs of numbers that are ε–indistinguishable, i.e. $R_\varepsilon = \{(x, x') \in X \times X \mid |x - x'| \leq \varepsilon\}$. Although we speak of ε–indistinguishability, the relation R_ε is not an equivalence relation, because it is not transitive. The non–transitivity is caused by the fact that x and x' as well as x' and x'' might be ε–indistinguishable, but x and x'' are not necessarily ε–indistinguishable, since $|x - x'| \leq \varepsilon$ and $|x' - x''| \leq \varepsilon$ does in general not imply $|x - x''| \leq \varepsilon$.

In practical applications it is often reasonable instead of fixing one value for ε to consider a whole range of values, for example the set $[0, 1]$. The interval $[0, 1]$ could be replaced by any other interval $[0, t]$. But since we will take scaling factors into account later on, it is sufficient to restrict ourselves to the unit interval. We define the mapping

$$E : X \times X \to [0,1], \quad (x,x') \mapsto 1 - \inf\{\varepsilon \in [0,1] \mid (x,x') \in R_\varepsilon\}. \tag{1}$$

The greater the value $E(x,x')$ the more x and x' are indistinguishable or similar. In this simple example we have $E(x,x') = 1 - \min\{|x - x'|, 1\}$. Obviously,

$$(x,x') \in R_\varepsilon \iff E(x,x') \geq 1 - \varepsilon \tag{2}$$

holds (compare also [20, 4]). In this sense, we can interpret the degree $E(x,x')$ of indistinguishability between x and x' in terms of an error or tolerance bound ε, i.e. $E(x,x')$ is greater than $1-\varepsilon$ if and only if x and x' are indistinguishable with respect to the tolerance bound ε. Furthermore, for a point $x_0 \in X$ we can describe the set of points that are ε–indistinguishable to x_0 by $[x_0 - \varepsilon, x_0 + \varepsilon] = \{x \in \mathbb{R} \mid E(x,x_0) \geq 1 - \varepsilon\}$. Intuitively, $E(x,x')$ can be read as the degree to which x and x' are equal or to which they can be identified. This motivates the following definition.

Definition 2.1 *A mapping $E : X \times X \to [0,1]$ is an* equality relation *if E satisfies the following axioms.*

(i) $E(x,x) = 1$

(ii) $E(x,x') = E(x',x)$

(iii) $E(x,x') + E(x',x'') - 1 \leq E(x,x'')$.

Condition (iii) is motivated by the fact that $(x,x') \in R_\varepsilon$ and $(x',x'') \in R_{\varepsilon'}$ implies $(x,x'') \in R_{\varepsilon+\varepsilon'}$. It can also be interpreted as a transitivity condition in the sense that if x and x' as well as x' and x'' are 'equal' to some degree, then x and x'' must also be 'equal' to some degree. Therefore, in the spirit of this notion of transitivity we could replace condition (iii) by

$$T(E(x,x'), E(x',x'')) \leq E(x,x'')$$

where $T(\alpha,\beta) = \min\{\alpha + \beta - 1, 1\}$ is the Łukasiewicz t–norm. From a theoretical point of view, any other t–norm is also reasonable, leading to a different notion of an equality relation. For such more foundational discussions on equality relations see for example [5, 8]. Equality relations are also called similarity relations [22] or (with arbitrary t–norms) indistinguishability operators (see for example [20]). For this paper it is sufficient to consider equality relations only as described in Definition 2.1.

Example 2.2 As we have already seen, when we have motivated equality relations, that the standard metric δ on $\mathbb{R}$ given by $\delta(x,x') = |x - x'|$ induces an equality relation E_δ via $E_\delta(x,x') = 1 - \min\{|x - x'|, 1\}$. More generally, if δ is a metric on the set X, then $E_\delta(x,x') = 1 - \min\{\delta(x,x'), 1\}$ is an equality relation on X. If, for example, $X = \mathbb{R}^n$, then we can take $\delta(x,x') = \| x - x' \|$ as an appropriate metric, where $\| . \|$ is a norm. □

Example 2.3 In Example 2.2 we considered the equality relation induced by the standard metric on $\mathbb{R}$. This metric is not always appropriate, since it does not take any scaling factors into account. For instance, if we want to compute the degree to which the ages of two persons are equal, the result depends on the unit (years, months, days, etc.) which we use for measuring the age. To amend this, we might introduce a scaling factor $c \geq 0$ and define the equality relation $E^{(c)}$

$$E^{(c)}(x, x') = 1 - \min\{|c \cdot x - c \cdot x'|, 1\}. \tag{3}$$

Depending on the unit we use for measuring, we have to choose an adequate scaling factor c. For example, in order to obtain the same indistinguishability that we have for the unit month when we measure in years, we would have to take the scaling factor 12 into account. □

Example 2.4 In some applications it is reasonable to modify the idea of a scaling factor as described in Example 2.3. Let us assume that we consider the real interval $[a, b]$ as possible values (our set for the vague environment). Instead of one scaling factor for all values, we can specify different scaling factors for different ranges.

A vague environment induced by a measuring instrument that works with high precision in the range of $[-1, 1] \subseteq [a, b]$ and with less precision outside this area, might be characterized by choosing the scaling factor 3 for the range $[-1, 1]$ and the scaling factor 0.5 for the area outside $[-1, 1]$. This means that we magnify the interval $[-1, 1]$ by the factor 3 and we shorten the intervals $[-1, a]$ and $[1, b]$ by the factor 2 (i.e. we 'magnify' them by the factor 0.5). To illustrate this idea, let us assume $a = -3$ and $b = 5$. In order to 'measure' the distance between two elements $x, x' \in [-3, 5]$, we map the interval $[-3, 5]$ (piecewise linearly) to the interval $[0, 9]$, where $[-3, -1]$, $[-1, 1]$, $[1, 5]$ is mapped linearly to $[0, 1]$, $[1, 7]$, $[7, 9]$, respectively. This piecewise linear transformation is given by the mapping

$$f : [-3, 5] \to [0, 9], \quad \mapsto \begin{cases} 0.5 \cdot (x + 3) & \text{if } -3 \leq x \leq -1 \\ 3 \cdot (x + 1) + 1 & \text{if } -1 \leq x \leq 1 \\ 0.5 \cdot (x - 1) + 7 & \text{if } 1 \leq x \leq 5. \end{cases}$$

The equality relation on $[-3, 5]$ induced by this transformation is defined by

$$E : [-3, 5] \times [-3, 5] \to [0, 1], \quad (x, x') \mapsto 1 - \min\{|f(x) - f(x')|, 1\}.$$

This equality relation is intended to model the vague environment induced by a measuring instrument that does not measure with the same exactness over the whole range $[-3, 5]$. Therefore, the equality relation reflects indistinguishability.

It is also reasonable to define such an equality relation to represent similarity. For example, if we want to regulate the room temperature, we may use a thermometer with a certain exactness for temperatures between $0°C$ and $35°C$. But for our purposes we are not interested in a precise value for a temperature below $15°C$ or above $27°C$, since we consider these temperatures as much too cold or warm, respectively, so that we have to heat or cool the room as much as possible. Temperatures between $15°C$ and $19°C$ or between $23°C$ and $27°C$ are also considered as too cold or too warm. But heating or cooling should be carried out moderately in these cases. For temperatures between $19°C$ and $23°C$ we are interested in more exact measurements, since these temperatures are near the optimal value for the room temperature and the adjustment has to be chosen carefully. In order to reflect this vague environment we might use the transformation

$$f : [0, 35] \to [0, 8], \quad \mapsto \begin{cases} 0 & \text{if} \quad 0 \leq x \leq 15 \\ 0.25 \cdot (x - 15) & \text{if} \quad 15 \leq x \leq 19 \\ 1.5 \cdot (x - 19) + 1 & \text{if} \quad 19 \leq x \leq 23 \\ 0.25 \cdot (x - 23) + 7 & \text{if} \quad 23 \leq x \leq 27 \\ 8 & \text{if} \quad 27 \leq x \leq 35, \end{cases}$$

leading to the equality relation $E(x, x') = 1 - \min\{|f(x) - f(x')|, 1\}$. □

Example 2.5 In Example 2.4 we considered vague environments where we specified different factors $c \geq 0$ for different intervals. A scaling factor $c > 1$ for an interval means that we 'look at this interval through a magnifying glass' and the indistinguishability or similarity between values of this interval is low. On the contrary, a scaling factor $c < 1$ implies that the values of this interval show great indistinguishability or similarity. We now consider a more general approach where we associate to each element of our vague environment a scaling factor, describing the magnifying factor with which we look at the neighbourhood of the element. If the interval $[a, b]$ is the underlying set of our vague environment, then we can represent the idea of scaling factors by a mapping $c : [a, b] \to [0, \infty)$. Assuming that the mapping c is integrable, the corresponding transformation is given by

$$f : [a, b] \to [0, \infty), \quad x \mapsto \int_a^x c(t)dt.$$

Again, we obtain the equality relation, characterizing our vague environment by

$$E : [a, b] \times [a, b] \to [0, 1], \quad (x, x') \mapsto 1 - \min\{|f(x) - f(x')|, 1\}.$$

□

The idea of equality relations (or indistinguishability operators) based on different scalings is also studied and characterized in terms of monotonic scaling mappings in [9, 11].

Example 2.6 In the above examples the underlying set of the vague environment was considered as a subset of the real numbers. This example is devoted to the vague environment with the set of fuzzy sets of X as the underlying set. Generally, two fuzzy sets $\mu, \nu : X \to [0, 1]$ are considered to be equal if and only if $\mu(x) = \nu(x)$ holds for all $x \in X$. But in some cases it might be reasonable to view two fuzzy sets whose membership functions are nearly identical as 'nearly equal' or similar although there exists an $x \in X$ such that $\mu(x) \neq \nu(x)$. The equality relation

$$E(\mu, \nu) = \inf_{x \in X} \{1 - |\mu(x) - \nu(x)|\}$$

on the set of fuzzy sets of X can be used to represent this idea of similarity between fuzzy sets. □

2.2 Points and Sets in Vague Environments

Operating in vague environments requires to reconsider concepts like points and sets. If we have to deal with a vague environment with indistinguishability, we have to take into account that a crisp value obtained from a measuring instrument does in general not correspond exactly to that value according to the indistinguishability. The same holds for the phenomenon of similarity. Specifying a crisp value x_0 in a vague environment generally refers only to *approximately* x_0. Thus, when speaking of crisp points or sets in a vague environment, we have to take into account the indistinguishability or similarity induced by that environment. Let us return to the simple representation of inexactness as it was considered in the beginning of this Section, i.e. we identify two values if their distance is less than a (fixed) value ε. In this case, the crisp value x_0 induces the interval $[x_0 - \varepsilon, x_0 + \varepsilon]$ – the set of points that are ε–indistinguishable to x_0. As we had already mentioned, we generally do not fix one value for ε, but consider a whole set of such tolerance bounds. A reasonable representation of what we associate with a crisp point x_0 would then be the set of intervals induced by varying the tolerance bound ε between 0 and 1, i.e. x_0 could be associated with the set

$$\{[x_0 - \varepsilon, x_0 + \varepsilon] \mid \varepsilon \in [0, 1]\}. \tag{4}$$

Of course, this representation is a bit clumsy. Therefore we make use of the equality relation E characterizing our vague environment X and associate with the crisp element $x_0 \in X$ the mapping

$$\mu_{x_0} : X \to [0, 1], \quad x \mapsto E(x_0, x), \tag{5}$$

specifying for each x to which degree x is equal to x_0. The mapping μ_{x_0} is a *fuzzy set*, which could be characterized as the fuzzy set of all $x \in X$ that are equal to x_0 with respect to the vague environment. In a vague environment each crisp point is associated with a fuzzy set. In this sense, we can interpret certain fuzzy sets as representations of crisp points in a vague environment described by an equality relation (which is not necessarily specified explicitly).

Definition 2.7 *Let E be an equality relation on the set X. For $x_0 \in X$ the fuzzy set μ_{x_0} given in* (5) *is called the* singleton *corresponding to x_0.*

Note that in the literature the term fuzzy singleton is often used for fuzzy sets that assign the membership degree 1 to exactly one point and 0 to all others (see for example [16]). This notion is not in contradiction to Definition 2.7. Since usually nothing is assumed about a vague environment, we might consider the (crisp) equality relation

$$E : X \times X \to [0,1], \qquad (x, x') \mapsto \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise.} \end{cases}$$

For this equality relation the singletons in the sense of Definition 2.7 are fuzzy singletons in the usual sense.

In the same way a crisp point leads to a fuzzy set in a vague environment, we have to associate with a crisp set the corresponding fuzzy set of all points that are 'equal' to one of the points in the crisp set. Analogously to the representation (4) for the point x_0 in a vague environment, we associate with a subset $M \subseteq X$ the set

$$\left\{ \bigcup_{x \in M} [x - \varepsilon, x + \varepsilon] \mid \varepsilon \in [0,1] \right\}.$$

$\bigcup_{x \in M} [x - \varepsilon, x + \varepsilon]$ is the set of points that are ε–indistinguishable to at least one of the elements in M. Again, this clumsy notation can be abbreviated in terms of the corresponding equality relation, leading to the following definition.

Definition 2.8 *Let E be an equality relation on the set X. For a set $M \subseteq X$ the fuzzy set*

$$\mu_M : X \to [0,1], \quad x \mapsto \sup\{E(m,x) \mid m \in M\}$$

is called the extensional hull *of M.*

Note that Definition 2.8 is a generalization of Definition 2.7, since $\mu_{x_0} = \mu_{\{x_0\}}$ holds.

Example 2.9 Let $X = \mathbb{R}$ and let $E = E_\delta$ be the equality relation induced by the standard metric as in Example 2.2. The singleton corresponding to $x_0 \in \mathbb{R}$ is the fuzzy set $\mu_{x_0}(x) = 1 - \min\{|x_0 - x|, 1\}$. Note that μ_{x_0} is a triangular membership function. □

Example 2.10 If we consider the interval $[a,b]$ instead of the point x_0 in Example 2.9, the extensional hull $\mu_{[a,b]}$ of $[a,b]$ is given by the trapezoidal membership function

$$\mu_{[a,b]}(x) = \begin{cases} 1 & \text{if } a \le x \le b \\ \max\{1 - a + x, 0\} & \text{if } x \le a \\ \max\{1 - x + b, 0\} & \text{if } b \le x \end{cases}$$

□

Triangular or trapezoidal membership functions with steeper or less steeper slopes can be obtained by using an equality relation with a scaling factor as described in Example 2.3. Examples 2.9 and 2.10 show that triangular or trapezoidal membership functions have a very appealing interpretation in the setting of vague environments. They can be justified from a theoretical point of view based on the concept of the canonical metric on the real numbers, not only as fuzzy sets that are easy to store and lead to simple computations. Note that, when we use the more general concept of a scaling function for the definition of an equality relation as in Example 2.5 the fuzzy sets obtained in this vague environment are in general no longer piecewise linear. Depending on the scaling function any type of convex, piecewise differentiable fuzzy set – for instance bell–shaped fuzzy sets – may appear as the extensional hull of a point.

In order to illustrate the concepts of singletons and extensional hulls let us consider the following example.

Example 2.11 Consider the set $X = \mathbb{Z} \times \mathbb{Z}$ of grid points as a subset of the Euclidean plane. Without taking any indistinguishabilities or similarities into account, the specification of one point $x_0 \in X$ intuitively corresponds to 'lifting or picking up' this point from the grid, while all other points stay in the plane.

But what happens if X is a vague environment? To illustrate that, we connect each grid point to the plane by an elastic rubber band. We also connect the neighbouring grid points by elastic rubber bands that reflect the degree of equality of the two neighbouring grid points, i.e. the less elastic the connecting rubber band is, the more are the two points 'equal'. If we now try to lift or pick up a single element of the grid other points will also be lifted up. The farther away they are from the chosen point, the less they are lifted up. What we obtain is a representation of the singleton (fuzzy set) representing the chosen point in the vague environment.

For the construction of the extensional hull we have to consider the same model. But instead of lifting one point, we pick up a set of points. □

2.3 Mappings

Until now we only considered one vague environment. Now we turn to the problem of handling more than one vague environment. Let us consider two vague environments X and Y which are characterized by the equality relations E and F, respectively. If φ is a mapping from X to Y, then φ should respect the equality relations in some sense. A reasonable requirement for φ is that, if x and x' are 'equal' to a certain degree in X, then the images $\varphi(x)$ and $\varphi(x')$ of these two elements should also be 'equal' to some degree in Y. This motivates the following definition.

Definition 2.12 *Let E and F be equality relations on X and Y, respectively. A mapping $\varphi : X \to Y$ is called* extensional *if*

$$E(x,x') \leq F\big(\varphi(x), \varphi(x')\big)$$

holds for all $x, x' \in X$.

The extensionality of a mapping requires that the degree to which the images of two elements are equal is not less than the degree to which the originals are equal. We can also interpret extensional mappings as error– or tolerance–preserving mappings, as shown in the following example.

Example 2.13 As we have seen in the beginning of Section 2.1, equality relations are related to the relations R_ε as described in (2), where the relation R_ε contains all pairs of elements that are ε–indistinguishable. ε can be interpreted as an error bound in the case of indistinguishability and as a tolerance bound in the case of similarity, in the sense that we identify two elements x and x' (with respect to R_ε) if $E(x,x') \geq 1-\varepsilon$. Or, in other words, if the equality relation E is defined as in (1), then x and x' are identified (with respect to R_ε) if the distance between them is less than or equal to the error– or tolerance bound ε. We denote the relations R_ε corresponding to the equality relations E and F by $R_\varepsilon^{(E)}$ and $R_\varepsilon^{(F)}$, respectively. Then the extensionality of the mapping $\varphi : X \to Y$ can be characterized in the following way.

$$\begin{aligned}
\varphi \text{ is extensional} \quad &\iff \quad \forall\ x, x' \in X : E(x,x') \leq F\big(\varphi(x), \varphi(x')\big) \\
&\iff \quad \forall\ x, x' \in X, \forall\ \varepsilon \in [0,1] : \\
&\qquad\quad \Big(E(x,x') \geq 1-\varepsilon \Longrightarrow F\big(\varphi(x), \varphi(x')\big) \geq 1-\varepsilon\Big) \\
&\iff \quad \forall\ x, x' \in X, \forall\ \varepsilon \in [0,1] : \\
&\qquad\quad \Big((x,x') \in R_\varepsilon^{(E)} \Longrightarrow \big(\varphi(x), \varphi(x')\big) \in R_\varepsilon^{(F)}\Big) \\
&\iff \quad \forall\ \varepsilon \in [0,1] : (\varphi \times \varphi)\big(R_\varepsilon^{(E)}\big) \subseteq R_\varepsilon^{(F)}. \qquad (6)
\end{aligned}$$

Considering ε as a tolerance– or error bound, (6) characterizes extensionality as tolerance– or error–bound–preserving. This means that if x and x' are identified, because the distance between them is less than the tolerance– or error bound ε, then also their images $\varphi(x)$ and $\varphi(x')$ are identified with respect to the same tolerance– or error–bound ε. □

Another problem that arises, when we have to take more than one vague environment into account, is to find an appropriate equality relation on the product space of the vague environments. The following Theorem proposes a solution to this problem.

Theorem 2.14 *Let $E_1, \ldots, E_n$ be equality relations on $X_1, \ldots, X_n$, respectively. Define*

$$\begin{aligned}
E_{1,\ldots,n} &: (X_1 \times \ldots \times X_n)^2 \to [0,1], \\
\big((x_1, \ldots, x_n), (x_1', \ldots, x_n')\big) &\mapsto \min\{E_1(x_1, x_1'), \ldots, E_n(x_n, x_n')\}. \qquad (7)
\end{aligned}$$

(a) $E_{1,\ldots,n}$ is an equality relation on $X_1 \times \ldots \times X_n$.

(b) The projection $\pi_i : X_1 \times \ldots \times X_n \to X_i$, $(x_1, \ldots, x_n) \mapsto x_i$ is extensional with respect to $E_{1,\ldots,n}$ and E_i for all $i \in \{1, \ldots, n\}$.

(c) *If E is an equality relation on $X_1 \times \ldots \times X_n$ such that all projections π_i ($i = 1, \ldots, n$) are extensional, then $E \leq E_{1,\ldots,n}$ holds.*

Proof.

(a) $E_{1,\ldots,n}$ obviously satisfies conditions (i) and (ii) of Definition 2.1. The third condition is also fulfilled, since

$$E_{1,\ldots,n}\big((x_1,\ldots,x_n),(x_1',\ldots,x_n')\big) + E_{1,\ldots,n}\big((x_1',\ldots,x_n'),(x_1'',\ldots,x_n'')\big) - 1$$

$$\begin{aligned}
&= \min_{i\in\{1,\ldots,n\}}\{E_i(x_i,x_i')\} + \min_{j\in\{1,\ldots,n\}}\{E_j(x_j',x_j'')\} - 1 \\
&\leq \min_{i\in\{1,\ldots,n\}}\{E_i(x_i,x_i') + E_i(x_i',x_i'') - 1\} \\
&\leq \min_{i\in\{1,\ldots,n\}}\{E_i(x_i,x_i'')\} \\
&= E_{1,\ldots,n}\big((x_1,\ldots,x_n),(x_1'',\ldots,x_n'')\big).
\end{aligned}$$

(b)

$$\begin{aligned}
E_{1,\ldots,n}\big((x_1,\ldots,x_n),(x_1',\ldots,x_n')\big) &\leq E_i(x_i,x_i') \\
&= E_i\big(\pi_i(x_1,\ldots,x_n),\pi_i(x_1',\ldots,x_n')\big).
\end{aligned}$$

(c) According to the extensionality of the projection π_i with respect to E' and E_i we obtain for all $i \in \{1,\ldots,n\}$

$$E'\big((x_1,\ldots,x_n),(x_1',\ldots,x_n')\big) \leq E_i(x_i,x_i'),$$

and therefore $E' \leq E_{1,\ldots,n}$. □

Theorem 2.14 characterizes the equality relation $E_{1,\ldots,n}$ as the coarsest (greatest) equality relation on $X_1 \times \ldots \times X_n$ such that all projections are extensional (tolerance– or error–bound–preserving). Of course, there are other equality relations on
$X_1 \times \ldots \times X_n$ making all projections extensional. In the case that the equality relations E_i are induced by the standard metric, it might be reasonable to consider other equality relations on the product space $X_1 \times \ldots \times X_n$ than (7). In the same way, a norm $\|\,.\,\|$ on the vector space $\mathbb{R}^n$ (inducing the metric δ by $\delta(x,x') = \|\, x - x' \,\|$) can be defined by

$$\|\,(x_1,\ldots,x_n)\,\| = \left(\sum_{i=1}^{n} |x_1|^p\right)^{1/p}, \tag{8}$$

we obtain an equality relation on the product space $X_1 \times \ldots \times X_n$ by

$$E\big((x_1,\ldots,x_n),(x_1',\ldots,x_n')\big) = 1 - \left(\sum_{i=1}^{n}(1 - E_i(x_i,x_i'))^p\right)^{1/p},$$

where $p \geq 1$ is a fixed constant. Using Minkowski's inequality it follows directly that E is an equality relation. Actually, E is the equality relation induced by the metric corresponding to the norm (8) in the sense of Example 2.2. Obviously, the projections are extensional with respect to E. Note that in the case of vector spaces often $p = 2$ is chosen, leading to the Euclidean norm.

2.4 Equality Relations Induced by Fuzzy Partitions

In Definition 2.7 we saw that certain fuzzy sets can be interpreted as representations of crisp points in a vague environment. As shown in Example 2.9 triangular membership functions are obtained, when the equality relation characterizing the vague environment is induced by the standard metric. A prerequisite for this interpretation of fuzzy sets is the specification of an appropriate equality relation in advance. Although it might be possible or even natural in some cases to specify an equality relation directly, the general methodology is to specify fuzzy sets without providing a corresponding equality relation. If we are given a set of fuzzy sets, we might suspect that the person who defined these fuzzy sets had an equality relation in mind (perhaps not consciously), and the fuzzy sets correspond to representations of crisp points in a vague environment. The problem we have to solve, is to find an appropriate equality relation. Formally, the given problem can be stated as follows.

> Let $(\mu_i)_{i \in I}$ be a family of normal fuzzy sets on X. $(\mu_i)_{i \in I}$ is also called a fuzzy partition. Let $(x_i)_{i \in I} \subseteq X$ be a family of elements of X such that $\mu_i(x_i) = 1$ for all $i \in I$.
> Is there an equality relation E on X such that $\mu_i = \mu_{x_i}$ for all $i \in I$, i.e. such that the fuzzy sets $(\mu_i)_{i \in I}$ are representations of the crisp elements $(x_i)_{i \in I}$ in the vague environment X?

Let us illustrate the problem by an example.

Example 2.15 Let $X = [-4, 4]$. Figure 1 shows a typical fuzzy partition. In this partition the five fuzzy sets $\mu_{\text{nb}}, \mu_{\text{ns}}, \mu_{\text{zero}}, \mu_{\text{ps}}, \mu_{\text{pb}}$ appear. We associate to these fuzzy sets the crisp values $x_{\text{nb}} = -4, x_{\text{ns}} = -2, x_{\text{zero}} = 0, x_{\text{ps}} = 2, x_{\text{pb}} = 4$, respectively. For each of these values the corresponding fuzzy set yields the membership degree 1.

It is easy to find an answer to the question if there is an equality relation on X such that the five fuzzy sets of the fuzzy partition can be interpreted as representations of the values $-4, -2, 0, 2, 4$ in the vague environment X. Recalling the Examples 2.3 and 2.9 we define the equality relation E on X by choosing $c = 0.5$ in equation (3). The equality relation E is not the only solution for this problem. We will give an example for another solution differing from E later on.

□

If the characterizing equality relation of a vague environment is not as simple as in Example 2.15, it might be very convenient to specify the fuzzy partition directly

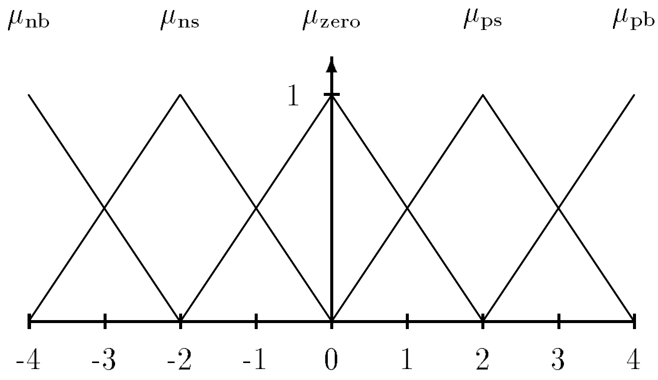

Figure 1: *A typical fuzzy partition.*

instead of defining the equality relation and choosing crisp points, that induce the fuzzy sets of the fuzzy partition in the form of singletons. A fuzzy set can be understood as a local specification of an equality relation in the neighbourhood of the element for which the membership degree is equal to 1.

We now answer the question, when the fuzzy sets of a fuzzy partition can be interpreted as representations of crisp points in a vague environment.

Theorem 2.16 *Let $(\mu_i)_{i\in I}$ be a family of normal fuzzy sets on X and let $(x_i)_{i\in I} \subseteq X$ be a family of elements of X such that $\mu_i(x_i) = 1$ for all $i \in I$. The following two statements are equivalent.*

(i) There exists an equality relation E on X such that the fuzzy sets μ_i correspond to the singletons induced by the elements x_i, i.e.

$$\mu_i = \mu_{x_i} \tag{9}$$

holds for all $i \in I$.

(ii) The inequality

$$\sup_{x\in X}\{\mu_i(x) + \mu_j(x) - 1\} \leq \inf_{y\in X}\{1 - |\mu_i(y) - \mu_j(y)|\} \tag{10}$$

holds for all $i, j \in I$.

Proof. $(i) \Rightarrow (ii)$. Let E be an equality relation satisfying (9). We prove (10) by showing that

$$\sup_{x\in X}\{\mu_i(x) + \mu_j(x) - 1\} \leq E(x_i, x_j) \tag{11}$$

and

$$E(x_i, x_j) \leq \inf_{y \in X}\{1 - |\mu_i(y) - \mu_j(y)|\} \tag{12}$$

are satisfied for all $i, j \in I$. From the assumption (9) we obtain

$$\begin{aligned}
\sup_{x \in X}\{\mu_i(x) + \mu_j(x) - 1\} &\leq \sup_{x \in X}\{E(x_i, x) + E(x_j, x) - 1\} \\
&\leq \sup_{x \in X}\{E(x_i, x_j)\} \\
&= E(x_i, x_j).
\end{aligned}$$

Thus (11) holds. We prove (12) by deriving the inequality

$$E(x_i, x_j) \leq 1 - |\mu_i(y) - \mu_j(y)|$$

for all $y \in X$. Without loss of generality let $E(x_i, y) \geq E(x_j, y)$. Taking (9) into account, we get

$$\begin{aligned}
E(x_i, x_j) &= 1 - E(x_i, y) + \big(E(x_i, y) + E(x_i, x_j) - 1\big) \\
&\leq 1 - E(x_i, y) + E(x_j, y) \\
&= 1 - |\mu_i(y) - \mu_j(y)|.
\end{aligned}$$

$(ii) \Rightarrow (i)$. Define

$$E : X \times X \to [0, 1], \qquad (x, x') \mapsto \inf_{i \in I}\{1 - |\mu_i(x) - \mu_i(x')|\}. \tag{13}$$

According to the representation theorem of Valverde [21], E is an 'indistinguishability operator' (with respect to the Łukasiewicz t–norm), which is nothing else than an equality relation in in the sense of Definition 2.1.

$\mu_i(x_i) = 1$ implies

$$\begin{aligned}
\mu_{x_i}(x) &= E(x_i, x) \\
&= \inf_{j \in I}\{1 - |\mu_j(x_i) - \mu_j(x)|\} \\
&\leq 1 - |\mu_i(x_i) - \mu_i(x)| \\
&= \mu_i(x).
\end{aligned}$$

What remains to be proved is $\mu_i(x) \leq \mu_{x_i}(x)$ for all $x \in X$. This means that we have to show that

$$\mu_i(x) \leq 1 - |\mu_j(x_i) - \mu_j(x)| \tag{14}$$

holds for all $j \in I$.

Let us firstly consider the case where $\mu_j(x_i) \leq \mu_j(x)$ is satisfied, implying that the right–hand side of (14) becomes

$$\begin{aligned}
1 - \mu_j(x) + \mu_j(x_i) &= 1 - |1 - \mu_j(x_i)| - \mu_j(x) + 1 \\
&= 1 - |\mu_i(x_i) - \mu_j(x_i)| - \mu_j(x) + 1
\end{aligned}$$

$$
\begin{aligned}
&\geq \inf_{y\in X}\{1-|\mu_i(y)-\mu_j(y)|\}-\mu_j(x)+1\\
&\overset{(10)}{\geq} \sup_{y\in X}\{\mu_i(y)+\mu_j(y)-1\}-\mu_j(x)+1\\
&\geq \mu_i(x)+\mu_j(x)-1-\mu_j(x)+1\\
&= \mu_i(x).
\end{aligned}
$$

For the case $\mu_j(x) \leq \mu_j(x_i)$ we obtain

$$
\begin{aligned}
1-\mu_j(x_i)+\mu_j(x) &= 1+\mu_j(x)-(\mu_j(x_i)+\mu_i(x_i)-1)\\
&\geq 1+\mu_j(x)-\sup_{y\in X}\{\mu_j(x)+\mu_i(x)-1\}\\
&\overset{(10)}{\geq} 1+\mu_j(x)-\inf_{y\in X}\{1-|\mu_i(y)-\mu_j(y)|\}\\
&\geq 1+\mu_j(x)-(1-|\mu_i(x)-\mu_j(x)|)\\
&= \mu_j(x)+|\mu_i(x)-\mu_j(x)|\\
&\geq \mu_i(x).
\end{aligned}
$$

□

The necessary and sufficient condition (10) for the existence of an equality relation under which the fuzzy sets of a fuzzy partition can be interpreted as representations of crisp points seems to be very technical. But we can provide an intuitive interpretation for it. We replace the left–hand side of inequality (10) by

$$
\sup_{x\in X}\{T(\mu_i(x),\mu_j(x))\} \qquad (15)
$$

where T is the Łukasiewicz t–norm. (15) yields the greatest degree to which an element of X belongs to the fuzzy set μ_i and to the fuzzy set μ_j, i.e. (15) is the degree to which μ_i and μ_j are not disjoint. Recalling Example 2.6, the right–hand side of (10) represents the degree to which μ_i and μ_j are equal. Therefore, all what 2.6 requires of the fuzzy partition is a disjointness condition, namely, that for each two fuzzy sets of the fuzzy partition their degree of non–disjointness must not exceed their degree of equality. For crisp sets this means nothing else than that two sets of a partition should either be disjoint or equal. The condition (10) and its interpretation was first discovered in [7]. It can be generalized to other t–norms [13].

Note that although condition (10) in Theorem 2.16 only guarantees the existence of an appropriate equality relation, one adequate equality relation is explicitly defined in the proof of Theorem 2.16 in equation (13). Of course, there is in general more than one equality relation such that the fuzzy sets correspond to singletons (see Example 2.18).

Theorem 2.16 describes a necessary and sufficient condition for the existence of an underlying equality relation for a fuzzy partition. Although we gave an intuitively appealing interpretation of this condition, it might be tedious to check its validity. The following Theorem provides a sufficient condition for the existence of an appropriate equality relation that is easily verified.

Theorem 2.17 *Let $(\mu_i)_{i\in I}$ be a family of normal fuzzy sets on X and let $(x_i)_{i\in I} \subseteq X$ be a family of elements of X such that $\mu_i(x_i) = 1$ for all $i \in I$. If*

$$\forall\, x \in X: \ \ \mu_i(x) + \mu_j(x) \leq 1 \tag{16}$$

holds for all $i \neq j$, then the fuzzy sets μ_i correspond to the singletons induced by the elements x_i with respect to the equality relation (13)*, i.e. $\mu_i = \mu_{x_i}$ holds for all $i \in I$.*

Proof. According to Theorem 2.16 it is sufficient to prove that condition (10) is satisfied. For $i = j$ the right–hand side of (10) yields 1, which is surely not less than the left–hand side. Taking (16) into account, the left–hand side of (10) is zero for $i \neq j$. □

Condition (16) can be rewritten as $T(\mu_i(x), \mu_j(x)) = 0$, where T is the Łukasiewicz t–norm, which means that the fuzzy sets μ_i and μ_j have to be disjoint with respect to the intersection induced by the Łukasiewicz t–norm. This disjointness condition is satisfied for many of the fuzzy partitions used in fuzzy control, where it is very often assumed that 'neighbouring' fuzzy sets meet at the height 0.5.

Example 2.18 We consider the fuzzy partition given in Figure 1, which obviously fulfills condition (16), so that Theorem 2.17 guarantees the existence of an equality relation such that the fuzzy sets correspond to the singletons induced by the points $-4, -2, 0, 2, 4$. This is not very surprising, since an appropriate equality relation was already specified in Example 2.15. Although this equality relation looks very natural, since it is induced by the standard metric (modulo the scaling factor 0.5) on $[-4, 4]$, it does not coincide with the equality relation E (see (13)) constructed in the proof of Theorem 2.16.

For $x \in \{-4, -2, 0, 2, 4\}$ and for all $x' \in X$ we obtain

$$E(x, x') = 1 - \min\{|0.5 \cdot x - 0.5 \cdot x'|, 1\}.$$

But we also have $E(-3, 3) = 0.5$ and $E(-3, 0) = 0$, which seems very peculiar, since $E(-3, 3) > E(-3, 0)$, in spite of $-3 < 0 < 3$. This paradoxical phenomenon can be explained, when we take the definition of E into account and derive

$$\begin{aligned} E(x, x') &\geq 1 - \sup_{i\in I}\big\{\max\{\mu_i(x), \mu_i(x')\}\big\} + \inf_{j\in I}\big\{\min\{\mu_i(x), \mu_i(x')\}\big\} \\ &\geq 1 - \sup_{i\in I}\big\{\max\{\mu_i(x), \mu_i(x')\}\big\}. \end{aligned} \tag{17}$$

The term (17) is simply the maximum of the degrees to which x and x' are covered by the fuzzy partition. This means that elements that are only 'partially' covered to the degree α have a degree of equality not less than $1 - \alpha$. For $x = -3$ and $x' = 3$ (17) yields for the fuzzy partition in Figure 1 the value 0.5.

If condition (9) of Theorem 2.16 is satisfied, then (13) is the coarsest (greatest) equality relation for which the fuzzy sets of the fuzzy partition can be understood as singletons. Therefore, for practical purposes one should use this equality relation only if a canonical equality relation as in Example 2.15 cannot be specified. □

3 Fuzzy Control Based on Vague Environments

Now that we are armed with the necessary background of vague environments provided in Section 2, we can apply these concepts to knowledge–based or fuzzy control.

3.1 Vague Environments and Mamdani's Model

Let us first describe the (simplified) formalization of the control problem that we want to consider here. We are given n input variables $\xi_1, \ldots, \xi_n$ taking values in the sets $X_1, \ldots, X_n$, respectively. For reasons of simplicity we assume that we have one output– or control variable η with values in the set Y. The problem we have to solve is to find an adequate control function $\varphi : X_1 \times \ldots \times X_n \to Y$, that assigns to each input tuple $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$ an appropriate output value $y = \varphi(x_1, \ldots, x_n) \in Y$.

Before we introduce knowledge–based control based on the notion of vague environments, we shortly recall how Mamdani's fuzzy control model [18] is defined, since we will see later on that we obtain also Mamdani's model as a result of our approach. The control function φ is specified by k linguistic control rules R_r in the form

$$R_r : \text{ if } \xi_1 \text{ is } A^{(1)}_{i_1,r} \text{ and } \ldots \text{ and } \xi_n \text{ is } A^{(n)}_{i_n,r} \text{ then } \eta \text{ is } B_{i_r} \qquad (r = 1, \ldots, k),$$

where each linguistic term $A^{(1)}_{i_1,r}$, …, $A^{(n)}_{i_n,r}$, B_{i_r} is associated to a fuzzy set $\mu^{(1)}_{i_1,r}$, …, $\mu^{(n)}_{i_n,r}$, μ_{i_r} on $X_1, \ldots, X_n, Y$, respectively. If we are given the input tuple $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$, the output of Mamdani's fuzzy controller is the fuzzy set

$$\mu^{\text{output}}_{x_1,\ldots,x_n} : Y \to [0,1], \qquad y \mapsto \max_{r \in \{1,\ldots,k\}} \left\{ \min\{\mu^{(1)}_{i_1,r}(x_1), \ldots, \mu^{(n)}_{i_n,r}(x_n), \mu_{i_r}(y)\} \right\}$$

on Y. In order to obtain a crisp output value, the fuzzy set $\mu^{\text{output}}_{x_1,\ldots,x_n}$ has to be defuzzified. A very common defuzzification strategy is the center of area method, but also the mean of maximum– and the max criterion method are applied (see for example [17]). For our purposes, it is sufficient to know that these defuzzification strategies compute a crisp value from a fuzzy set, the exact algorithm for each method is not of importance for this paper.

We now come to the presentation of a concept of knowledge–based control based on vague environments, which looks at first glance completely different from Mamdani's model, although there are parallels to the rationale behind Zadeh's compositional rule of inference [23, 24]. But it turns out that the same computations are carried out. Thus we are able to provide a reasonable semantics for Mamdani's model.

The first step in the design of a controller based on vague environments is the specification of appropriate equality relations $E_1, \dots, E_n, F$ on the sets $X_1, \dots, X_n, Y$, respectively. These equality relations are intended to model the indistinguishability or similarity of values as described in Section 2. Remember that there are two different concepts of indistinguishability. In fuzzy control we mainly have to deal with intended indistinguishability, which is not enforced by difficulties in measuring exact values, but which is intended to model the fact that arbitrary precision is not needed. Later on we can make use of this fact, since it will be sufficient to specify controller outputs only for certain 'typical' values. Taking the (intended) indistinguishability into account, we can extend this partially defined control function to a fully defined one.

It is possible to define the equality relations representing the indistinguishability directly, but a more convenient way is to define them as in the Examples 2.2 – 2.5. Especially scaling functions are very appealing, since they have a reasonable interpretation. Small scaling factors imply a low distinguishability, meaning that in this area the control action changes only slowly with varying input values. A greater scaling factor indicates a high distinguishability, i.e. even small variations of the inputs might lead to greater alterations in the control action.

In the next step a control expert has to provide a set of input–output tuples, i.e. tuples $\big((x_1^{(r)}, \dots, x_n^{(r)}), y^{(r)}\big) \in (X_1 \times \dots \times X_n) \times Y$ $(r = 1, \dots, k)$. The tuple $\big((x_1^{(r)}, \dots, x_n^{(r)}), y^{(r)}\big)$ simply means that $y^{(r)}$ is the appropriate output value for input $(x_1^{(r)}, \dots, x_n^{(r)})$. These k input–output tuples correspond to a partial specification of the control function, since they can be understood as the function

$$\varphi_0 : \big\{(x_1^{(r)}, \dots, x_n^{(r)}) \mid r \in \{1, \dots, k\}\big\} \to Y, \quad (x_1^{(r)}, \dots, x_n^{(r)}) \mapsto y^{(r)},$$

which is a partial mapping from $X_1 \times \dots \times X_n$ to Y.

Our task is now to determine an appropriate output value $\varphi(x_1, \dots, x_n) = y \in Y$ for an arbitrary input $(x_1, \dots, x_n) \in X_1 \times \dots \times X_n$ from the knowledge given by the equality relations and the partial control function. In order to solve this problem and to illustrate our approach, we recall, how we can derive the value $y = \varphi(x)$ for an ordinary mapping $\varphi : X \to Y$ by drawing the graph G_φ of φ. We obtain $y = \varphi(x)$ by projecting the graph at x to the set Y. The method is shown in Figure 2.

Unfortunately, instead of the control function φ we only know the partial mapping φ_0 and are in the unlucky situation illustrated in Figure 3. But fortunately, Figure 3 does not contain all available information, since it shows only the specified input–output tuples without taking the equality relations on $X_1, \dots, X_n$, and Y into account. Instead of the crisp graph G_{φ_0} of the partial mapping φ_0 we should consider the extensional hull of G_{φ_0} according to Definition 2.8 and Examples 2.9 – 2.11. Before we can compute this fuzzy set, representing the extensional hull, we have to combine the equality relations defined on $X_1, \dots, X_n$, and Y to an equality

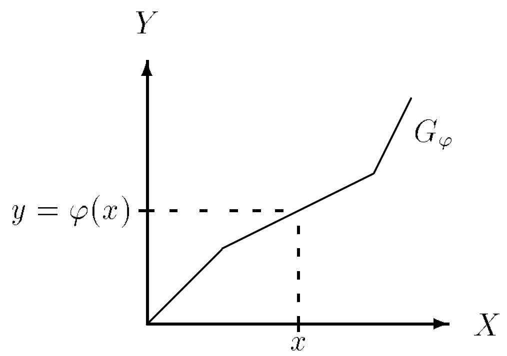

Figure 2: *Obtaining* $y = \varphi(x)$ *from the graph* G_φ *of* φ.

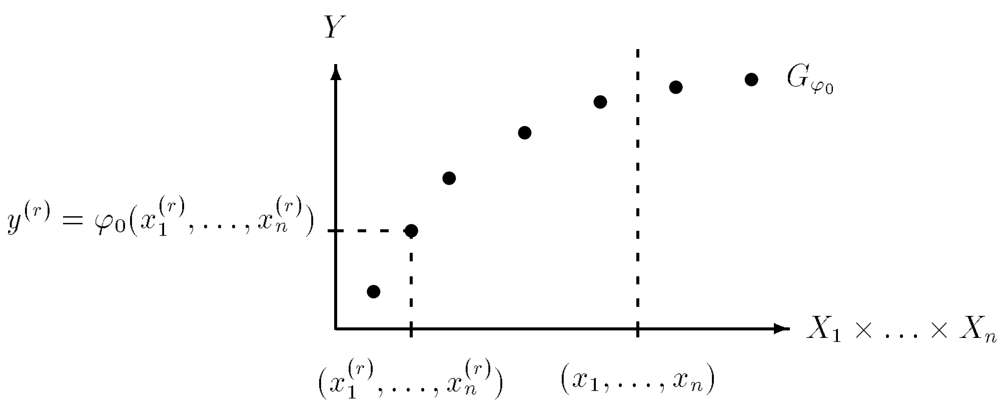

Figure 3: $y = \varphi_0(x_1, \ldots, x_n)$ *is not defined.*

relation E on $X_1 \times \ldots \times X_n \times Y$. Being cautious, we recall Theorem 2.14 and choose the coarsest equality relation on $X_1 \times \ldots \times X_n \times Y$ such that all projections are extensional, i.e.

$$E\big((x_1, \ldots, x_n, y), (x_1', \ldots, x_n', y')\big) = \min\{E_1(x_1, x_1'), \ldots, E_n(x_n, x_n'), F(y, y')\}.$$

The extensional hull of the graph G_{φ_0} with respect to this equality relation is, according to Definition 2.8, the fuzzy set

$$\mu_{G_{\varphi_0}}(x_1, \ldots, x_n, y) = \max_{r \in \{1,\ldots,k\}} \left\{ E\big((x_1^{(r)}, \ldots, x_n^{(r)}, y^{(r)}), (x_1, \ldots, x_n, y)\big) \right\}$$

on $X_1 \times \ldots \times X_n \times Y$. To obtain an 'output' for a given input tuple $(x_1, \ldots, x_n)$ we compute the projection not of the graph G_{φ_0}, but of its extensional hull at $(x_1, \ldots, x_n)$, which leads to the fuzzy set

$$\mu_{\varphi_0}^{(x_1,\ldots,x_n)}(y) = \max_{r \in \{1,\ldots,k\}} \left\{ \min\{E_1(x_1^{(r)}, x_1), \ldots, E_n(x_n^{(r)}, x_n), F(y^{(r)}, y)\} \right\} \tag{18}$$

on Y. Remembering Definition 2.7 we can rewrite the fuzzy set (18) in the form

$$\mu_{\varphi_0}^{(x_1,\ldots,x_n)}(y) = \max_{r \in \{1,\ldots,k\}} \left\{ \min\{\mu_{x_1^{(r)}}(x_1), \ldots, \mu_{x_n^{(r)}}(x_n), \mu_{y^{(r)}}(y)\} \right\}, \tag{19}$$

since $E_1(x_1^{(r)}, \,.\,), \ldots, E_n(x_n^{(r)}, \,.\,), F(y^{(r)}, \,.\,)$ corresponds to the singleton (fuzzy set) $\mu_{x_1^{(r)}}, \ldots, \mu_{x_n^{(r)}}, \mu_{y^{(r)}}$, respectively.

Now we are able to see the connection to Mamdani's model. For our approach we started with the specification of equality relations on the sets $X_1, \ldots, X_n, Y$ and a partial control mapping $\varphi_0 : X_1 \times \ldots \times X_n \to Y$ in the form of the set

$$\left\{ \big((x_1^{(r)}, \ldots, x_n^{(r)}), y^{(r)}\big) \mid r \in \{1, \ldots, k\} \right\}. \tag{20}$$

Taking into account that due to the equality relations we are working in vague environments, this partial mapping can be interpreted as k control rules of the form

$$\begin{array}{lll} R_r: & \text{if } \xi_1 \text{ is } \textit{approximately } x_1^{(r)} \text{ and } \ldots \text{ and } \xi_n \text{ is } \textit{approximately } x_n^{(r)} & (21) \\ & \text{then } \eta \text{ is } \textit{approximately } y^{(r)} & (r = 1, \ldots, k) \end{array}$$

where *approximately* $x_1^{(r)}$, …, *approximately* $x_n^{(r)}$, *approximately* $y^{(r)}$ is represented by the singleton (fuzzy set) $\mu_{x_1^{(r)}}, \ldots, \mu_{x_n^{(r)}}, \mu_{y^{(r)}}$, respectively. Taking the above control rules together with these fuzzy sets, we can define a fuzzy controller in the sense of Mamdani and obtain for the input $(x_1, \ldots, x_n)$ the fuzzy set $\mu_{x_1,\ldots,x_n}^{\text{output}}$ as output. This fuzzy set coincides with the fuzzy set $\mu_{\varphi_0}^{(x_1,\ldots,x_n)}$, which is the output derived in our knowledge–based control model based on vague environments. Let us put this remarkable fact into a Theorem.

Theorem 3.1 *Let $E_1, \ldots, E_n, F$ be equality relations on $X_1, \ldots, X_n, Y$, respectively. Furthermore, let $\varphi_0 : X_1 \times \ldots \times X_n \to Y$ be a partial (control) mapping given by the set* (20). *From this we obtain a fuzzy controller in the sense of Mamdani by taking* (21) *as control rules where we associate to the linguistic terms* approximately $x_1^{(r)}$, *…,* approximately $x_n^{(r)}$, approximately $y^{(r)}$ *the fuzzy sets $\mu_{x_1^{(r)}}, \ldots, \mu_{x_n^{(r)}}, \mu_{y^{(r)}}$, respectively.*

Then the output fuzzy set $\mu^{\text{output}}_{x_1,\ldots,x_n}$ in Mamdani's model coincides with the output fuzzy set $\mu_{\varphi_0}^{(x_1,\ldots,x_n)}$ of the controller based on vague environments.

Theorem 3.1 states that we can translate our control approach to Mamdani's model and obtain in both models the same output (before defuzzification).

The obvious question that turns up is, whether a fuzzy controller in the sense of Mamdani can be translated to a controller based on vague environments. The answer is yes if the fuzzy partitions used in Mamdani's model satisfy condition (10) of Theorem 2.16 or the stronger, but easier to check condition (16) of Theorem 2.17, which is fulfilled in many applications. In this case, we can derive equality relations induced by the fuzzy partitions, for example by using (13). The fuzzy sets of the fuzzy partitions then correspond to the singletons induced by the elements where they reach the membership degree 1. The partial control mapping φ_0 can be derived from the control rules in Mamdani's model by replacing the linguistic terms by the elements that induce the corresponding fuzzy sets as singletons.

3.2 Extensional Control Mappings

In the previous Subsection we have provided a knowledge–based control model based on vague environments, that can be translated to Mamdani's model. The pre-requisite for this approach is the specification of appropriate equality relations and of a partial control mapping. By considering the extensional hull of the graph of the partial control mapping we were able to derive an output fuzzy set for each input. In this subsection we discard this graph theoretical view and catch on the idea of an extensional (control) function. Starting from equality relations and a partial control mapping φ_0, we look for a control function $\varphi : X_1 \times \ldots \times X_n \to Y$ that is

(i) identical with φ_0 for those elements for which φ_0 is defined

(ii) extensional with respect to the equality relation E on $X_1 \times \ldots \times X_n$ and F on Y where E is the equality relation given by formula (7) induced by the equality relations $E_1, \ldots, E_n$ on $X_1, \ldots, X_n$.

Condition (i) is an obvious requirement for the control function. (ii) is a very strict claim. But it can be justified by interpreting extensional mappings as tolerance– or error bound preserving as in Example 2.13. If a control function is not extensional, this means that there are input values wich are highly indistinguishable. But their corresponding outputs differ significantly and are highly distinguishable. This is not coherent with the idea that the vague environments reflect the indistinguishability inherent in the control prblem. If we do not distinguish very well between two input values, their corresponding outputs should also be almost similar. In this sense extensionality is a reasonable concept.

The two conditions (i) and (ii) do in general not determine a unique control function. They provide only a criterion to evaluate a given control function. A possibility to define a control function is for example to choose a defuzzification strategy for Mamdani's or for our control model based on vague environments. Applying the defuzzification strategy to the fuzzy set $\mu^{\text{output}}_{x_1,\ldots,x_n}$ or $\mu^{(x_1,\ldots,x_n)}_{\varphi_0}$, respectively, yields for each input $(x_1,\ldots,x_n)$ a crisp output $y \in Y$ and therefore determines a control function.

Let us now investigate the restrictions that the extensionality requirement imposes on the control function φ. We consider a controller based on vague environments with the partial control mapping given by (20) and the equality relations $E_1,\ldots,E_n,F$ on $X_1,\ldots,X_n,Y$, respectively.

Let $(x_1,\ldots,x_n) \in X_1 \times \ldots \times X_n$ be an input tuple. In order to be extensional φ has at least to satisfy the condition

$$E\big((x_1^{(r)},\ldots,x_n^{(r)}),(x_1,\ldots,x_n)\big) \le F\big(y^{(r)},\varphi(x_1,\ldots,x_n)\big). \tag{22}$$

for all $r \in \{1,\ldots,k\}$. Recalling the definition of the equality relation E and taking into account that we can rewrite $E_i(x_1^{(r)},\,.\,)$ $(i = 1,\ldots,n)$ and $F(y^{(r)},\,.\,)$ as the singletons (fuzzy sets) $\mu_{x_i^{(r)}}$ and $\mu_{y^{(r)}}$, respectively, (22) translates to

$$\begin{aligned}\min_{i\in\{1,\ldots,n\}}\{E_i(x_i^{(r)},x_i)\} &= \min_{i\in\{1,\ldots,n\}}\{\mu_{x_i^{(r)}}(x_i)\}\\ &\le F(y^{(r)},(\varphi(x_1,\ldots,x_n))\\ &= \mu_{y^{(r)}}(\varphi(x_1,\ldots,x_n)).\end{aligned} \tag{23}$$

Let us abbreviate

$$\alpha_r = \min_{i\in\{1,\ldots,n\}}\{\mu_{x_i^{(r)}}(x_i)\}.$$

If we consider the translation of our controller based on vague environments to Mamdani's model, then α_r is the degree to which the antecedent of the rule R_r is satisfied. (23) requires to choose $\varphi(x_1,\ldots,x_n)$ from the α_r–cut of the fuzzy set $\mu_{y^{(r)}}$. Since (23) has to be satisfied for all $r \in \{1,\ldots,k\}$, $\varphi(x_1,\ldots,x_n)$ should be in the intersection of the α_r–cuts of the fuzzy sets $\mu_{y^{(r)}}$. This intersection is illustrated in Figure 4.

The requirement of an extensional control function φ which coincides with the partial control mapping φ_0 for those elements for which φ_0 is defined is a very restrictive condition. In some cases such a function does not exist at all, for example, if the intersection indicated in Figure 4 is empty. Even if such a mapping exists, it is in general not uniquely determined, nor is it easy to find. What we obtain from the conditions (i) and (ii) is a criterion to evaluate a given control function. If a control function is not extensional, it should not necessarily be rejected, but it might be unreasonable to have a control function that is far from being extensional.

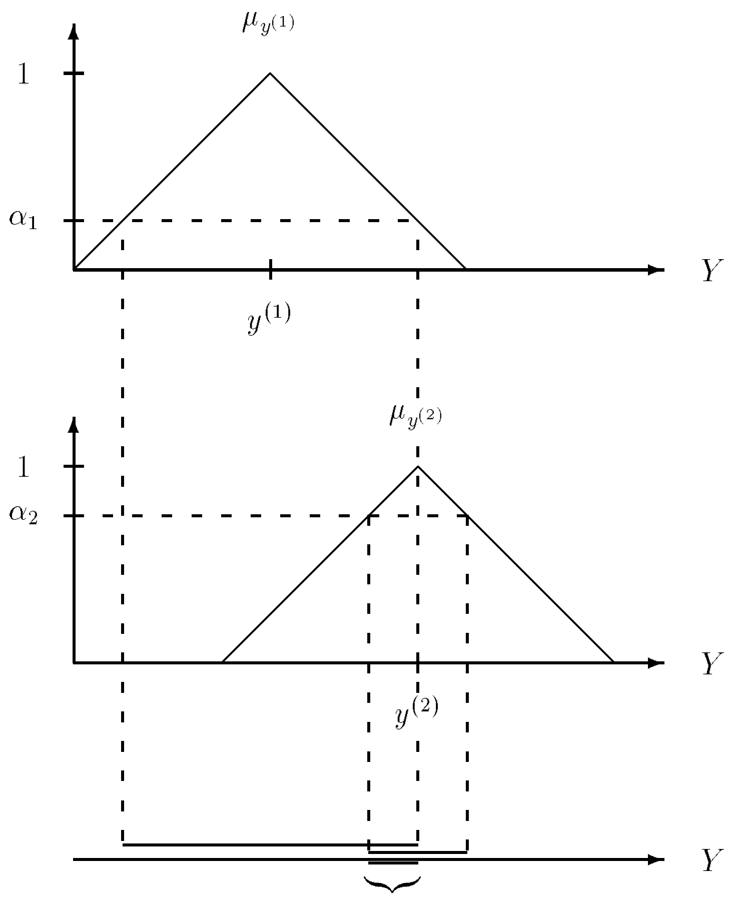

Figure 4: *For φ to be extensional, $\varphi(x_1, \ldots, x_n)$ has to be chosen from the indicated set.*

Note that the extensionality requirement for the control function can also be stated in Mamdani's model, since we have seen in the previous Subsection that we can translate Mamdani's model to our approach if the fuzzy partitions in Mamdani's model satisfy condition (10) of Theorem 2.16.

4 Design of a Fuzzy Controller in Vague Environments

In this Section we summarize the results of Section 3 by investigating the methodology of Mamdani's model and our approach. The first decision, the designer of a fuzzy controller has to make, is which model he prefers. We consider only Mamdani's and our approach here. Let us first describe the tasks to be carried out when applying our concept of vague environments.

4.1 Fuzzy Control Based on Equality Relations

The control expert has to specify appropriate equality relations for the sets of values for the input– and output variables. These equality relations are intended to reflect indistinguishability (according to problems in obtaining exact measurements or data) and similarity (modelling the idea that the control expert does not always need high precision values). Note that we usually have to deal with similarity, when we want to establish a knowledge–based controller.

It is not recommendable to try to define an equality relation directly. A very convenient and reasonable way to specify an equality relation is provided by the methods proposed in the Examples 2.3 – 2.5. The scaling factors used in these Examples have a canonical interpretation in the sense that they characterize how strong (scaling factor < 1) or how week (scaling factor > 1) the indistinguishability or similarity in a certain region is. In applications it is reasonable to choose a large scaling factor in the range where the process is very sensitive to small changes. In ranges where great tolerance for the output value is acceptable a low scaling factor is adequate (compare Example 2.4). Of course, the first choice of the scaling factors and the ranges where they are valid might not be optimal, and a fine tuning later on in order to obtain the desired control actions should be considered.

In addition to the equality relations we need a partial control mapping that assigns appropriate outputs to certain inputs. This partial control mapping should be defined by the experienced control expert or it can be derived from data gained from observing a control expert.

To make use of the equality relations and the partial control mapping for a controller, we have to decide, whether we accept the graph theoretical view for determining a control function as we have presented it in Subsection 3.1. This graph theoretical view provides for each input $(x_1, \ldots, x_n)$ the fuzzy set $\mu_{\varphi_0}^{(x_1,\ldots,x_n)}$ as output by considering the extensional hull of the graph of the partial control mapping and projecting it at $(x_1, \ldots, x_n)$ to the set of possible output values. A defuzzification strategy is necessary to obtain a crisp output value.

If the very restrictive condition of extensionality for the control function is required, then the condition (23) illustrated in Figure 4 should be satisfied by the control function induced by the defuzzification strategy. If the graph theoretical view and a defuzzification strategy are not accepted, the restrictions imposed by condition (23) can be used to define a control function. An extensional control function does not necessarily exist and extensionality does not lead to a unique control function in general. It is also reasonable to judge a control function by considering how far it is from being extensional, i.e. how strong it violates condition (23).

4.2 Fuzzy Control Based on Fuzzy Sets

In Mamdani's model instead of equality relations and a partial control mapping fuzzy partitions and a rule base have to be specified. Mamdani's model can stand for itself alone, but we can also translate it to our approach. From this translation we might get some insights about good or bad behaviour of the controller induced by Mamdani's model.

In order to translate Mamdani's model to our approach we have to compute the equality relations from the given fuzzy partitions. Corresponding equality relations exist if condition (10) of Theorem 2.16 or the stronger requirement (16) of Theorem 2.17 is satisfied for the fuzzy partitions. In this case, the equality relations can be defined by equation (13), which leads to the coarsest solution for the equality relations. As pointed out in Example 2.18, there might be other appropriate equality relations, that can be defined as in Example 2.18. The control expert should check the equality relations and decide whether they reflect the indistinguishabilities and similarities he has in mind for the process. A rejection of the equality relations can lead to a revision of the fuzzy partitions.

When satisfactory equality relations are derived from the fuzzy partitions, the partial control mapping can be defined taking the linguistic control rules into account. Since the fuzzy sets in Mamdani's model correspond to singletons in our approach, we can associate with each fuzzy set the crisp value which induces the fuzzy set as the corresponding singleton. In this way, a rule translates to an 'input–output' tuple where each linguistic term is replaced by the crisp value that is associated with the fuzzy set representing the linguistic term. The control expert should judge whether the partial control mapping derived from the rules looks reasonable.

If extensionality of the control function is desired, then condition (23) has to be satisfied by the control function induced by the defuzzification strategy. If the extensionality condition is not fulfilled, then either the defuzzification strategy, the rules (the partial control mapping), or the fuzzy partitions (equality relations) have to be modified.

5 Application to Engine Idle Speed Control

In the previous Sections we introduced a new theoretical and semantic approach to fuzzy control. It has to be checked, whether the basic concept of equality relations and the presented results are appropriate for solving existing control problems of industrial practice.

For this reason in cooperation between Volkswagen AG, Wolfsburg, Germany, and the Department of Computer Science of the University of Braunschweig a fuzzy idle speed controller for the Volkswagen 2000cc 116hp spark ignition engine of the Golf GTI model has been developed.

The controller is based on an analysis of the underlying motor process and consists of a Mamdani fuzzy controller which is embedded in a so–called "meta–controller". It turns out that the fuzzy controller has a better performance than the corresponding production–line controller. Therefore the new methods will also be applied to idle speed control problems with respect to other spark ignition engines of Volkswagen AG and Audi AG, respectively.

5.1 A Short Introduction to Idle Speed Control

Nowadays the intended performance standards of spark ignition engines make it necessary to decrease fuel consumption and pollutant emission. One specific problem in this field is the reduction of the engine idle speed, since the increased application of luxury equipment that raises motor load, like, for example, air–conditioning system or power steering, require a flexible control mechanism, since decreased RPM can lead to critical speed drops.

Generally there are two different kinds of engine idle speed control:

- Spark advance
- Volumetric control

In order to be most effective, our fuzzy controller only realizes volumetric control, while the sparking advance of the production–line car was retained.

The principle of volumetric control is shown in Figure 5.

On the existing vehicle a sudden drop of engine speed may have various reasons:

- switching on electrical units, which put an additional load on the motor via the three–phase alterator
- switching on the air–conditioning system, which puts an additional load on the motor via the air–conditioning compressor (including the additional cooling fan)
- activation of power steering, which adds the hydraulic circuit pump to the motor load

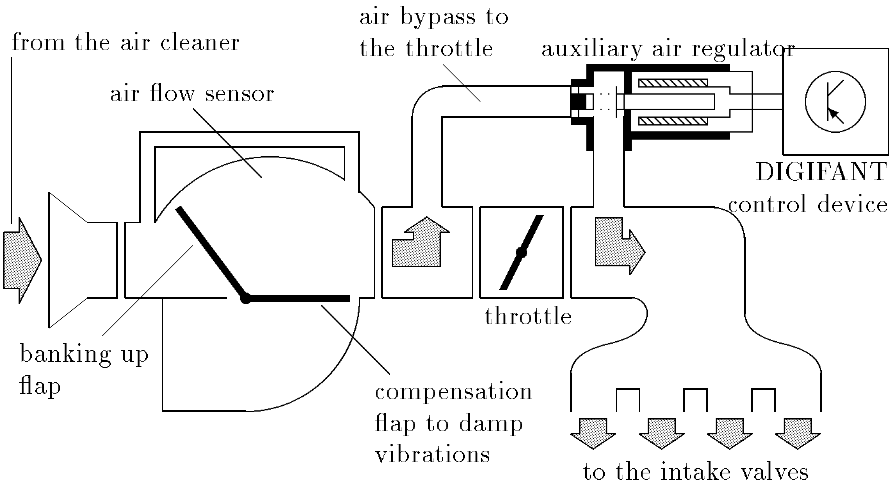

Figure 5: Principle of idle speed control

The task of volumetric control is to compensate the speed drop by increasing the cross–section. In this case the number or revolutions should get the target rotation speed as correct and as fast as possible.

One of the main problems is the low torque in range of low revolutions, because in extreme situations, like simultaneous switching–on of the air–conditioning system and the power steering, very strong and rapidly occurring speed drops can occur.

The above–mentioned lacking quality of engine speed information refers to the imprecision of the available signal of the Hall–pick–up in the ignition distributor (i.e. differences in the number of revolutions up to $\pm$ 30 rpm result from manufacturing tolerances, gear clearances and torsional vibrations).

Another problem arises from the plurality of additional stochastic processes in the system. As an example, bad combustions or deviations of ignition and fuel–injection have to be mentioned, because in this case a volumetric controller should not react on a differing engine speed in order to prevent an oscillation of the idle speed.

However, the delay of the automatic control system turns out to be the hardest problem, since it passes about ten sequences of ignition after changing the fuel inlet, until the motor delivers the changed torque (essential reason: air transfer time).

5.2 Design of the Fuzzy Controller

In order to solve the problems of engine idle speed control mentioned in Section 5.1, we chose an integrated conception of two controllers, where a Mamdani fuzzy controller (MFC) forms the basic unit, embedded in a meta–controller (Figure 6).

The fuzzy controller is implemented as a C program on a 386–processor laptop. The data communication takes place through the control device of the engine management system. The inputs are the number of revolutions REV0_LO, the tarfuzzy controller

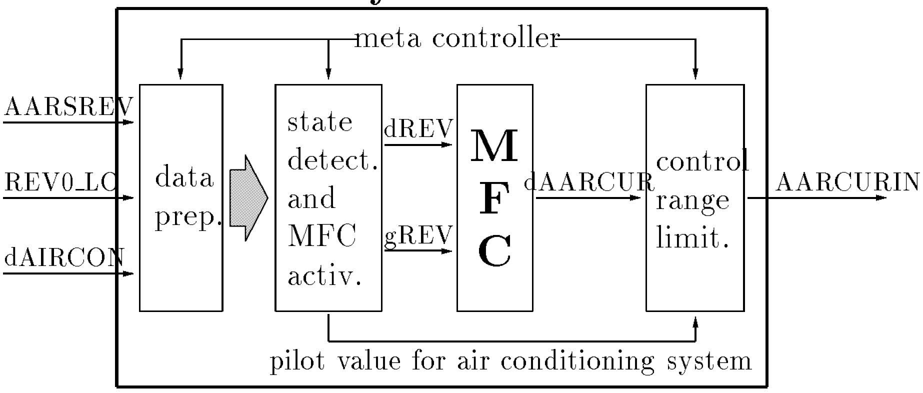

Figure 6: Structure of the fuzzy controller

get rotation speed AARSEV and the state flag dAIRCON of the air–conditioning system (on/off).

The electrical current AARCURIN for the auxiliary air regulator serves as output.

The meta–controller consists of three components: data modification, state detection (including MFC activation) and security stage. The data modification computes the mean of the incoming noisy data and supplies engine speed information for evaluation. If required, the state detection activates the MFC, and a new control value is determined using the modified and the original data.

A security stage behind the MFC takes care of limiting the control range.

At the time of switching on the air–conditioning system, the MFC is not activated, and a pilot output value has to be chosen to get the best control action.

The deviation dREV of the number of revolutions to the target rotation speed and the gradient gREV of the number of revolutions (to be understood as the difference of numbers of revolutions w.r.t. two measurement points) are the two input variables. The change of current dAARCUR for the auxiliary air regulator serves as the output variable.

5.3 How to Develop the Controller by Equality Relations

With respect to the general problems of engine idle speed control mentioned in Section 5.1, from a physical modeling point of view, the underlying motor process seems to be intractable. Furthermore, our experience with motor engineers has shown that even experts in their field are often not in the position to deliver at least a restricted description of the motor process by linguistic control rules.

For this reason the analysis was based on measurement data obtained by idle speed experiments with a real spark ignition engine of the considered type.

The analysis of the available data resulted in the specification of equality relations

E_ν ($\nu \in \{\text{dREV}, \text{gREV}, \text{dAARCUR}\}$) on the domains $X^{(\nu)}$, induced by integrable mappings $c_\nu : X^{(\nu)} \to [0, \infty)$, as we have already motivated them in Example 2.5.

The underlying domains are $X^{(\text{dREV})} = [-70, 70]$ (rotations per minute), $X^{(\text{gREV})} = [-40, 40]$ (rotations per minute), and $Y^{(\text{dAARCUR})} = [-25, 25]$, where the latter one is to be interpreted as a linear transformation of the real value of the current change dAARCUR.

The scaling function c_{gREV} is, for instance, defined as follows:

$$c_{\text{gREV}} : X^{(\text{gREV})} \to [0, \infty),$$

$$x \mapsto \begin{cases} \frac{1}{33}, & \text{if } -40 \leq x < -7 \\ 0, & \text{if } -7 \leq x < -4 \\ 1, & \text{if } -4 \leq x < -2 \\ 0, & \text{if } -2 \leq x < 2 \\ 1, & \text{if } 2 \leq x < 4 \\ 0, & \text{if } 4 \leq x < 7 \\ \frac{1}{33}, & \text{if } 7 \leq x < 40. \end{cases}$$

The basic principle for obtaining c_ν is to fix appropriate partitions of the sets $X^{(\nu)}$, and to calculate constant scaling factors for them.

In this connection it should be emphasized that the imprecision of the measured dREV values suggests the choice of minor distinguishability in an environment of 0 in order to avoid control actions that refer to stochastic error processes rather than to important state changes.

The next step of the development of our fuzzy controller is the partial specification $\varphi_0 : X^{(\text{dREV})} \times X^{(\text{gREV})} \to Y^{(\text{dAARCUR})}$ of the control mapping, which is also computed by the given experimental data, and shown in Table 1.

		gREV						
	φ_0	-40	-6	-3	0	3	6	40
dREV	-70	20	15	15	10	10	5	5
	-50	20	15	10	10	10	5	0
	-30	15	10	5	5	5	0	0
	0	5	5	0	0	0	-10	-5
	30	0	0	0	-5	-5	-10	-15
	50	0	-5	-5	-10	-15	-15	-20
	70	-5	-5	-10	-15	-15	-15	-15

Table 1: *The partial mapping φ_0 for idle speed control*

Then, with the aid of the equality relations E_ν and the partial mapping φ_0 a fuzzy controller can be constructed. Figure 7 illustrates the induced look–up table, where the selected defuzzification method is COA (center of area).

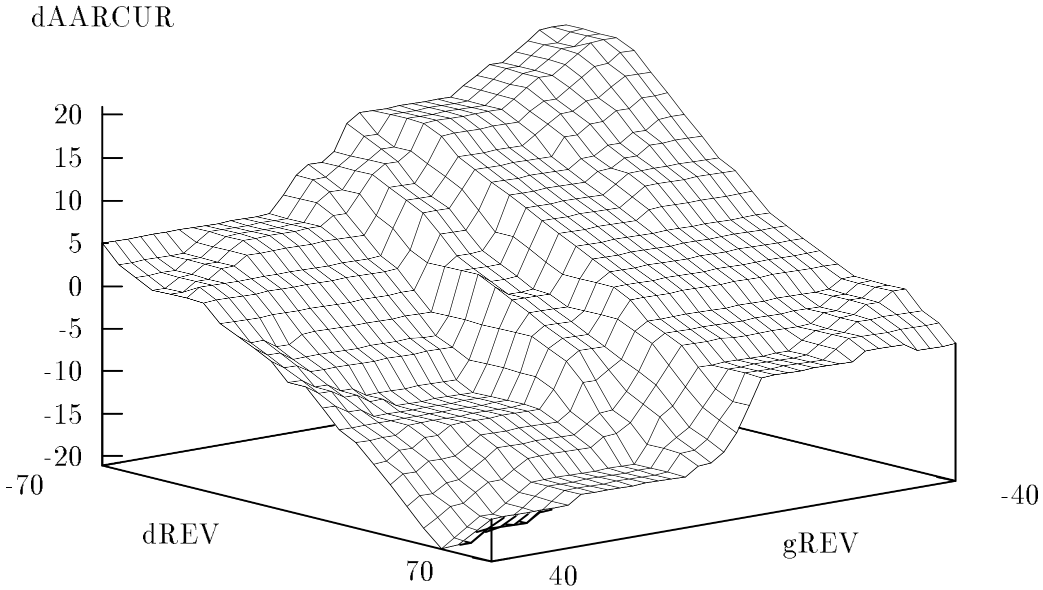

Figure 7: *Performance characteristics of the MFC*

Based on our presented new methodology of creating fuzzy controllers by the concept of equality relations, a promising final step of the development is to transfer the obtained controller into a Mamdani fuzzy controller, applying the technique described in Section 3.1. The main advantage of this transformation is the opportunity of using the available standard tools for an efficient implementation of Mamdani fuzzy controllers. In our example the resulting singletons are associated with linguistic terms negative big (nb), negative medium (nm), negative small (ns), approximately zero (az), positive small (ps), positive medium (pm), and positive big (pb), respectively.

Figure 8 illustrates the corresponding partition of the set $X^{(\mathrm{gREV})}$.

Note that in this application there is no extensional control function φ which coincides with the partial control mapping φ_0 for those elements for which φ_0 is defined.

But non–extensionality is acceptable in this case (comp. Section 3.2), since it only arises from slightly modified control values in order to avoid vibration phenomena in extreme situations.

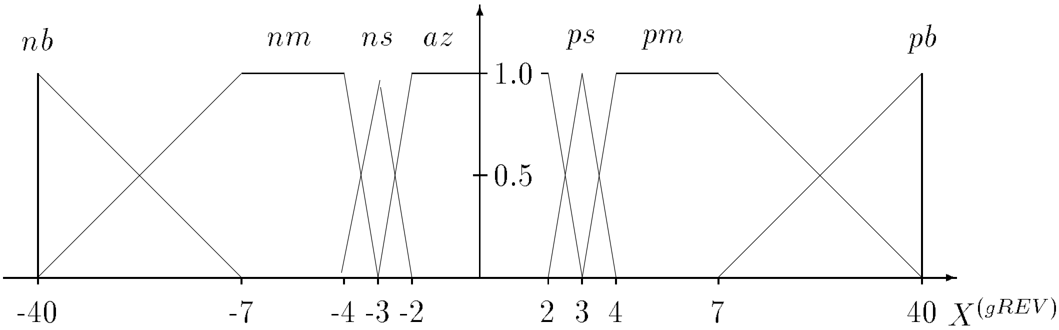

Figure 8: : *The fuzzy partition of* $X^{(gREV)}$

A comparison between the production–line controller and the fuzzy controller in real situations showed that the fuzzy controller has superior performance characteristics.

6 Conclusions

Fuzzy control in vague environments is based on the idea of interpolating a partially specified control function by exploiting the information hidden in the indistinguishability or similarity in the vague environments. It should be emphasized that the control rules of a fuzzy controller are not interpreted as logical implications, but as vague descriptions of crisp control actions in crisp situations. This is in accordance with the methods applied in fuzzy control where generally disjunctive instead of conjunctive rule bases are considered.

Our approach provides a semantical background for fuzzy control, which elucidates the concepts of fuzzy control and provides a clear explanation in which way the fuzzy sets and the control rules can be interpreted. The insights gained by our model can be used to understand and to carry out the design and the tuning of a fuzzy controller as described in Section 4. Although we have only discussed the connections of our model of fuzzy control in vague environments to Mamdani's fuzzy control approach, there are also relations between vague environments and the Sugeno–type fuzzy controller [19], namely that we have to consider the input space as a vague environment, whereas we deal with a crisp output space in the case of the Sugeno–type fuzzy controller.

We have interpreted equality relations in a very narrow sense, mainly in terms of a nearness measure, in order to simplify and clarify their meaning in the setting of fuzzy control, since in control applications the underlying sets are usually subsets of the real numbers. Of course, all other more general interpretations of equality relations in terms of indistinguishability operators or similarity relations are also admissible.

The justification we gave for Mamdani's fuzzy control model does not provide a concrete defuzzification strategy. Only in the case of extensional control functions certain restrictions have to be satisfied by the defuzzification strategy. Defuzzification methods, although intuitively compelling, are still not examined in a rigorous theoretical framework.

The aim of this paper is to provide a formal framework for a controller based on knowledge–based interpolation. As one of our results we obtain a reformulation of Mamdani's fuzzy control model with vague environments as a background so that we gain insight in this methodology and can use this insight for fuzzy controller design, tuning, and also for the development of learnable or adaptive fuzzy controllers.

Further research will be directed to control theoretic aspects of fuzzy control in vague environments like stability analysis or robustness. Taking the vague environments into account, it should be possible to modify these notions accordingly.

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