The Role of Similarity in Fuzzy Reasoning

Frank Klawonn
Department of Electrical Engineering
and Computer Science
FH Ostfriesland
Constantiaplatz 4
D-26723 Emden, Germany
e-mail klawonn@et-inf.fho-emden.de

Abstract

Fuzzy reasoning mechanisms are designed to cope with vague and uncertain knowledge and information. In this paper we demonstrate that from the vagueness inherent in a fuzzy system a canonical indistinguishability of objects can be derived which cannot be overcome by the standard reasoning schemes. We discuss also the consequences for fuzzy logic in the narrow sense.

1 Introduction

The fundamental concept in fuzzy systems is the notion of membership degree, generalizing from the idea of an element having crisp membership to a set, to gradual membership. Therefore, most fuzzy systems are based on a fuzzification of the predicate $\in$ (*is element of*). Another concept closely related to gradual membership is similarity or indistinguishability which may be modelled as a fuzzification of equality. A formalization of this notion in the general framework of GL-monoids is given in Section 2. The unit interval endowed with the usual ordering and a continuous t-norm is a special example of a GL-monoid. We prefer the more general notion of GL-monoids since in this context the fundamental concept – namely residuation which establishes a connection between many-valued conjunctions and implications – becomes more obvious than in the unit interval with its rich structure. Section 3 reviews some results on the indistinguishability inherent in standard approximate reasoning schemes that can be formalized in terms of fuzzy relations. Finally, in Section 4 these results are discussed in the view of fuzzy logic in the narrow sense.

2 GL-Monoids and Fuzzy Equality

As already mentioned in the introduction, we use GL-monoids as the formal framework for our investigations instead of the unit interval. From [4], we recall the definition of a GL-monoid.

Definition 1 *$(L, \leq, *)$ is a GL-monoid if*

1. *$(L, \leq)$ is a complete lattice,*
2. *$(L, *, \mathbb{1}, \mathbb{0})$ is a commutative monoid with unit $\mathbb{1}$ and zero element $\mathbb{0}$, i.e. the operation $* : L \times L \longrightarrow L$ is associative and commutative and the equations $\mathbb{1} * \alpha = \alpha$ and $\alpha * \mathbb{0} = \mathbb{0}$ hold for $\alpha \in L$,*
3. *$*$ is isotone, i.e.*
$$\alpha \leq \beta \implies \alpha * \gamma \leq \beta * \gamma,$$
4. *$(L, \leq, *)$ is integral, i.e. $\mathbb{1} = \bigvee L$*
5. *$(L, \leq, *)$ is the dual of a divisibility monoid, i.e.*
$$\alpha \leq \beta \text{ implies the existence of } \gamma \in L \text{ such that } \alpha = \beta * \gamma,$$
6. *$(L, \leq, *)$ is residuated, meaning that there exists a binary operation $\rightarrow$ on L satisfying*
$$\alpha * \beta \leq \gamma \iff \alpha \leq \beta \rightarrow \gamma, \tag{1}$$
7. *the infinite distributive law holds, i.e.*
$$\alpha * \bigvee_{i \in I} \beta_i \;=\; \bigvee_{i \in I} (\alpha * \beta_i).$$

Unless otherwise stated we assume $(L, \leq, *)$ to be a GL-Monoid. L will be considered as the set of 'truth' values of a many-valued logic. $\vee$ can be considered as the valuation function of a disjunction, $\wedge$ and $*$ as two alternatives for a conjunction. The binary operation $\rightarrow$ is uniquely determined by the adjunction property (1):

$$\alpha \rightarrow \beta \;=\; \bigvee \{\lambda \in L \mid \alpha * \lambda \leq \beta\}. \tag{2}$$

$\rightarrow$ can be viewed as the valuation function for the implication (associated with the conjunction $*$). From this implication we can derive in a canonical way a valuation for the negation by defining $\neg\alpha = \alpha \rightarrow \mathbb{0}$. Note that in a GL-monoid the zero element of $*$ is also the universal lower bound, i.e. $\mathbb{0} = \bigwedge L$. We define the biimplication $\leftrightarrow : L \times L \longrightarrow L$ by

$$\alpha \leftrightarrow \beta \;=\; (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha).$$

For a more detailed discussion of GL-monoids and their properties see [4, 8] and [5], which is devoted to the relation between logical calculi and structures like GL-monoids.

Interesting for applications is the case when L is the unit interval with the usual linear ordering. $*$ can be any continuous t-norm (a commutative, associative, non-decreasing binary operation on $[0,1]$ having 1 as unit). $*$ is understood as an alternative to the lattice operation $\wedge$ (in the case of the unit interval simply min) for the valuation function of a conjunction.

Example 1 Let $L = [0,1]$ be the unit interval with the usual ordering. Then

$$\alpha \leftrightarrow \beta \ = \ \max\{\alpha,\beta\} \to \min\{\alpha,\beta\}$$

holds [23]. It is easy to check that based on the choice of $*$ the following formula can be derived for $\to, \leftrightarrow$, and $\neg$ (cf. [7, 11]).

$\alpha * \beta$	$\max\{\alpha+\beta-1,0\}$	$\min\{\alpha,\beta\}$	$\alpha \cdot \beta$
$\alpha \to \beta$	$\min\{1-\alpha+\beta,1\}$	$\begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha \leq \beta \\ \frac{\beta}{\alpha} & \text{otherwise} \end{cases}$
$\alpha \leftrightarrow \beta$	$1 - \lvert\alpha - \beta\rvert$	$\begin{cases} 1 & \text{if } \alpha = \beta \\ \min\{\alpha,\beta\} & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha = \beta \\ \frac{\min\{\alpha,\beta\}}{\max\{\alpha,\beta\}} & \text{otherwise} \end{cases}$
$\neg\alpha$	$1 - \alpha$	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$

Interpreting L as the set of truth values, an L-fuzzy (sub)set (or simply a fuzzy set) of the set X is a mapping $\mu : X \longrightarrow L$. The value $\mu(x) \in L$ is understood as the degree or truth value of x being an element of the (sub)set μ.

Definition 2 *An equality relation (w.r.t. the operation $*$) on the set X is a mapping $E : X \times X \longrightarrow L$ satisfying the axioms:*

$$\begin{array}{lll} \text{(E1)} & E(x,x) = 1, & \text{(reflexivity)} \\ \text{(E2)} & E(x,y) = E(y,x), & \text{(symmetry)} \\ \text{(E3)} & E(x,y) * E(y,z) \leq E(x,z). & \text{(transitivity)} \end{array}$$

In the unit interval, depending on the choice of the operation $*$, sometimes E is also called a similarity relation [25, 15], indistinguishability operator [22], fuzzy equality (relation) [6, 9], fuzzy equivalence relation [21] or proximity relation [2]. Although these different names are used for the same concept, the underlying philosophy is the same, namely to have a notion that certain objects may be identified to a certain degree.

Considering the relation *element of* ($\in$) for ordinary sets, equal elements may be exchanged, i.e. we have

$$x \in M \text{ and } x = y \;\Rightarrow\; y \in M. \tag{3}$$

Replacing the crisp equality in this statement by an equality relation and the notion of a set by fuzzy set, we obtain the following definition.

Definition 3 *A fuzzy set $\mu \in L^X$ is called extensional w.r.t. the equality relation E on X iff*

$$\mu(x) * E(x,y) \;\leq\; \mu(y)$$

holds for all $x, y \in X$.

The fuzzy set

$$\hat{\mu} \;=\; \bigwedge\{\nu \mid \mu \leq \nu \text{ and } \nu \text{ is extensional w.r.t. } E\}$$

is called the extensional hull of μ w.r.t. E.

Obviously, an extensional fuzzy set coincides with its extensional hull and the extensional hull has the following properties.

(i) $\hat{\mu}(x) \;=\; \bigvee\{\mu(y) * E(x,y) \mid y \in X\}$,

(ii) $\hat{\mu}$ is extensional w.r.t. E,

(iii) $\hat{\hat{\mu}} = \hat{\mu}$.

It should be noted that a fuzzy set $\mu \in L^X$ is extensional w.r.t. the equality relation E if and only if

$$\mu(x) \leftrightarrow \mu(y) \;\geq\; E(x,y) \tag{4}$$

holds for all $x, y \in X$.

Note that for any fuzzy set μ we can define an equality relation $E_\mu(x,y) = \mu(x) \leftrightarrow \mu(y)$, having the property that it is the coarsest equality relation for which μ is extensional. Theorem 1 in the following section will provide a more general result.

3 Equality Relations in Approximate Reasoning

After we have introduced the notions of equality relations and extensional hulls in the previous section, we present some results that demonstrate the relevance of equality relations in approximate reasoning, in the sense that they characterize the indistinguishability inherent in a fuzzy system. For proofs and more details we refer to [8]. The equality relation that is defined in the following theorem will be of great importance for the rest of this paper.

Theorem 1 *Let $\mathcal{F} \subseteq L^X$ be a set of fuzzy sets. Then*

$$E_{\mathcal{F}}(x, y) = \bigwedge_{\mu \in \mathcal{F}} (\mu(x) \leftrightarrow \mu(y)) \qquad (5)$$

is the coarsest (greatest) equality relation on X such that all fuzzy sets in $\mathcal{F}$ are extensional w.r.t. $E_{\mathcal{F}}$.

The equality relation (5) already appeared in Valverde's representation theorem [24] which he proved for $L = [0, 1]$. This theorem states that $E_{\mathcal{F}}$ is an equality relation if and only if there is a set $\mathcal{F}$ of fuzzy sets such that E can be written in the form (5).

In [8] it was proved that the set $\mathcal{A}_E$ of all fuzzy sets that are extensional w.r.t. the equality relation E has the following closure properties. For any $\mathcal{B} \subseteq \mathcal{A}_E$, $\mu \in \mathcal{A}_E$, and $\alpha \in L$ we have:

(a) $(\bigvee \mathcal{B}) \in \mathcal{A}_E$,

(b) $(\bigwedge \mathcal{B}) \in \mathcal{A}_E$,

(c) $(\alpha * \mu) \in \mathcal{A}_E$,

(d) $(\mu \to \alpha) \in \mathcal{A}_E$,

(e) $(\alpha \to \mu) \in \mathcal{A}_E$.

Vice versa, for a set $\mathcal{A}$ of fuzzy sets fulfilling these axioms there exists a unique equality relation – namely the one given in Theorem 1 – such that $\mathcal{A}$ coincides with the set of all extensional fuzzy sets w.r.t. this equality relation. It is interesting to remark that, because an equality relation is uniquely determined by its set of extensional fuzzy sets, Valverde's representation theorem is also valid in the more general context of GL-monoids.

The above mentioned properties characterize equality relations in an algebraic sense. Another interesting approach is described in [16, 17, 18, 19, 20, 21] where connections between equality relations (or related concepts) and fuzzy partitions or fuzzy coverings establish are provided.

In approximate reasoning if-then rules of the form

$$\text{If } \xi \text{ is } A, \text{ then } \eta \text{ is } B, \qquad (6)$$

are very common where ξ and η are variables with domains X and Y, respectively. A and B are linguistic terms like *positive big* or *approximately zero* (see, e.g., [10]). These linguistic terms are usually modelled by suitable fuzzy sets, say $\mu_A \in L^X$ and $\mu_B \in L^Y$. In addition to such general rules one has specific information like

$$\xi \text{ is } A' \qquad (7)$$

where A' is represented by the fuzzy set $\mu_{A'} \in L^X$ (or simply by $\mu \in L^X$).

The application of a single rule of the form (6) to the information (7) is usually formalized on the basis of a computing scheme of the following form. The rule is encoded as a fuzzy relation of the form

$$\varrho(x, y) \;=\; \varrho_{\odot}(x, y) \;=\; \mu_A(x) \odot \mu_B(y) \tag{8}$$

where $\odot \in \{\wedge, *, \rightarrow\}$. For a given input information in the form of the fuzzy set $\mu_{A'} \in L^X$, the 'output' fuzzy set $\nu_{\text{conclusion}}$ is computed as the composition of the fuzzy relation $\varrho_{\odot}$ and the fuzzy set $\mu_{A'}$, i.e.

$$(\mu_{A'} \circ_{\sqcap} \varrho)(y) \;=\; \bigvee_{x \in X} \{\mu_{A'}(x) \sqcap \varrho(x, y)\} \tag{9}$$

for all $y \in Y$, where $\sqcap \in \{\wedge, *\}$ (cf., e.g., [1, 3, 10]). This scheme is called sup-$\wedge$-inference. In fuzzy control, for instance, usually $\sqcap = \min = \odot$ is chosen.

The following two theorems show that for such typical inference schemes the indistinguishability inherent in the fuzzy sets cannot be overcome.

Theorem 2 *Let $\mu, \mu_A \in L^X$, $\mu_B \in L^Y$. Furthermore, let E be an equality relation on X such that μ_A is extensional w.r.t. E. Let $\varrho_{\odot}$ be defined as in Equation* (8)*. Then for the combinations $\odot = \rightarrow$ and $\sqcap = *$, $\odot = *$ and $\sqcap = *$, $\odot = \wedge$ and $\sqcap = *$, the equation (cf. Equation* (9)*)*

$$(\mu \circ_{\sqcap} \varrho_{\odot}) \;=\; (\hat{\mu} \circ_{\sqcap} \varrho_{\odot})$$

is valid.

When we interpret Theorem 2 in the sense that the fuzzy sets μ_A and μ_B represent the linguistic terms A and B of an if-then rule of the form (6), then it states that for the mentioned combinations of operations for a given input μ the output fuzzy set $\mu \circ_{\sqcap} \varrho_{\odot}$ inferred by the rule does not change when we replace μ by its extensional hull. Although not explicitely mentioned, the case $\sqcap = \wedge$ is also included in the theorem, namely when we choose $* = \wedge$ for our GL-monoid.

For the output fuzzy sets we have an analogous result, namely, that the output fuzzy set is always extensional.

Theorem 3 *Let $\mu, \mu_A \in L^X$, $\mu_B \in L^Y$. Furthermore, let F be an equality relation on Y such that μ_B is extensional w.r.t. F. Let $\varrho_{\odot}$ be defined as in Equation* (8)*. Then for the combinations $\odot = \rightarrow$ and $\sqcap = *$, $\odot = *$ and $\sqcap = *$, $\odot = \wedge$ and $\sqcap = *$, the fuzzy set $(\mu \circ_{\sqcap} \varrho_{\odot})$ (cf. Equation* (9)*) is extensional w.r.t. F.*

The results of Theorems 2 and 3 can be easily extended to a set of if-then rules of the form

$$\text{If } \xi \text{ is } A_i, \text{ then } \eta \text{ is } B_i, \qquad (i \in I),$$

where the linguistic terms A_i and B_i are modelled by the fuzzy set $\mu_{A_i} \in L^X$ and $\mu_{B_i} \in L^Y$. The output fuzzy set for a given 'input fuzzy set' $\mu \in L^X$ is usually computed either by

$$\bigwedge_{i\in I} (\mu \circ_{\sqcap} \varrho_i) \tag{10}$$

or

$$\bigvee_{i\in I} (\mu \circ_{\sqcap} \varrho_i). \tag{11}$$

This does neither effect Theorem 2 nor Theorem 3 according to the closure properties (a) and (b). The theorems are also valid for fuzzy rules with more than one premise using the Cartesian product of equality relations (combining them by the minimum). For details see [8].

4 Equality Relations and Fuzzy Logic

In this section we extend the results derived in the previous section to fuzzy logic in the narrow sense. It would lead us astray to give a complete definition of first order fuzzy logic. A thorough introduction to this topic can be found in [12, 13]. What we mainly need to know for the context in which we consider fuzzy logic here is that fuzzy logic admits by truth values evaluated logical formulae. We concentrate on the predicates which correspond to fuzzy sets, i.e., an n-ary predicate is associated with a fuzzy set on X^n, when X is the underlying domain for variables. The question that we will examine is how well can objects in X be distinguished, when we consider a set of elementary predicates and take all predicates into account that can be formulated using the elementary predicates and the logical connectives and quantifiers. Formally, the question can be formulated in the following way. We are given a set $\mathcal{A}$ (the fuzzy sets associated with the elementary predicates) of fuzzy sets. What is the coarsest equality relation such that all fuzzy sets that fuzzy sets that can be defined with the fuzzy sets in $\mathcal{A}$ and the logical connectives are extensional?

Let us first restrict to unary predicates. Thus $\mathcal{A} \subseteq L^X$. The set of fuzzy sets we can build from $\mathcal{A}$ with the logical connectives is the smallest set $\mathcal{A}^*$ satisfying

(i) $\mathcal{A} \subseteq \mathcal{A}^*$

(ii) $\mathcal{A}_0 \subseteq \mathcal{A}^* \Rightarrow (\bigwedge \mathcal{A}_0) \in \mathcal{A}^*$ and $(\bigvee \mathcal{A}_0) \in \mathcal{A}^*$

(iii) $\mu \in \mathcal{A}^*, \alpha \in L \Rightarrow (\alpha \to \mu) \in \mathcal{A}^*$ and $(\mu \to \alpha) \in \mathcal{A}^*$

(iv) $\mu, \nu \in \mathcal{A}^* \Rightarrow (\mu \to \nu) \in \mathcal{A}^*$ and $(\mu * \nu) \in \mathcal{A}^*$

Note that the extensionality of μ and ν does in general not imply the extensionality of $\mu \to \nu$ or $\mu * \nu$. Thus the coarsest equality relation making all fuzzy sets in $\mathcal{A}^*$ extensional will be finer than the coarsest one making all fuzzy sets in $\mathcal{A}$ extensional.

Theorem 4 *Let $\mathcal{A} \subseteq L^X$ be a set of fuzzy sets and let $\mathcal{A}^* \subseteq L^X$ be the smallest set of fuzzy sets satisfying the above mentioned properties (i)-(iv). Then*

$$E_{\mathcal{A}^*}(x,y) \;=\; \bigwedge_{k \in \mathbb{N}} (E_{\mathcal{A}}(x,y))^k \tag{12}$$

holds for all $x, y \in X$. The exponent k is meant w.r.t. the operation $*$.

Proof. Let $E(x,y)$ denote the right hand side of equation (12). We prove that E is an equality relation making all fuzzy sets in $\mathcal{A}^*$ extensional which is not smaller than $E_{\mathcal{A}^*}$. This implies that E is equal to $E_{\mathcal{A}^*}$, since $E_{\mathcal{A}^*}$ is the coarsest equality relation making all fuzzy sets in $\mathcal{A}^*$ extensional.

An important property which we need in the proof is that in a GL-monoid we have that the idempotency of an element $\alpha \in L$, i.e. $\alpha * \alpha = \alpha$, implies $\alpha * \beta = \alpha \wedge \beta$ for all $\beta \in L$. (For a proof of this fact see [4].) From the definition of E it is clear that for all $x, y \in L$, $E(x,y)$ is an idempotent element of L.

E is obviously reflexive and symmetric. Making use of the idempotency of the values that E takes, we can prove that E is not only transitive w.r.t. $*$ but even w.r.t. $\wedge$.

$$\left(\bigwedge_{k \in \mathbb{N}} (E_{\mathcal{A}}(x,y))^k\right) \wedge \left(\bigwedge_{m \in \mathbb{N}} (E_{\mathcal{A}}(y,z))^m\right)$$

$$\begin{aligned}
&= \left(\bigwedge_{k \in \mathbb{N}} (E_{\mathcal{A}}(x,y))^k\right) * \left(\bigwedge_{m \in \mathbb{N}} (E_{\mathcal{A}}(y,z))^m\right) \\
&\leq \left(\bigwedge_{k,m \in \mathbb{N}} (E_{\mathcal{A}}(x,y))^k * (E_{\mathcal{A}}(y,z))^m\right) \\
&\leq \left(\bigwedge_{k \in \mathbb{N}} (E_{\mathcal{A}}(x,y))^k * (E_{\mathcal{A}}(y,z))^k\right) \\
&\leq \bigwedge_{k \in \mathbb{N}} (E_{\mathcal{A}}(x,z))^k .
\end{aligned}$$

Thus E is also transitive.

Define

$$\mathcal{B} \;=\; \{\mu \in L^X \mid \mu \text{ is extensional w.r.t. } E\}.$$

$\mathcal{B}$ is closed under the closure properties (a)-(d) mentioned in the previous section. But $\mathcal{B}$ is also closed w.r.t. the operations $\rightarrow$ and $*$. To see this, let $\mu, \nu \in \mathcal{B}$. Making again use of the idempotency of $E(x, y)$, we derive

$$\begin{array}{rcl}\mu(x) * \nu(x) * E(x,y) & = & \mu(x) * \nu(x) * E(x,y) * E(x,y) \\ & \leq & \mu(y) * \nu(y),\end{array}$$

i.e., E is closed w.r.t. $*$.

To prove the extensionality of $\mu \rightarrow \nu$, i.e.

$$(\mu(x) \rightarrow \nu(x)) * E(x,y) \;\leq\; \mu(y) \rightarrow \nu(y),$$

it is sufficient to prove

$$\mu(y) * (\mu(x) \rightarrow \nu(x)) * E(x,y) \;\leq\; \nu(y)$$

according to the residuation property.

$$\begin{array}{rcl}\mu(y) * (\mu(x) \rightarrow \nu(x)) * E(x,y) & = & E(x,y) * \mu(y) * (\mu(x) \rightarrow \nu(x)) * E(x,y) \\ & \leq & \mu(x) * (\mu(x) \rightarrow \nu(x)) * E(x,y) \\ & = & (\mu(x) \wedge \nu(x)) * E(x,y) \\ & \leq & \nu(x) * E(x,y) \\ & \leq & \nu(y),\end{array}$$

where we have made use of the property $\alpha * (\alpha \rightarrow \beta) = \alpha \wedge \beta$ (see [4]).

Since $E(x, y) \leq E_{\mathcal{A}}(x, y)$ holds, any fuzzy set that is extensional w.r.t. $E_{\mathcal{A}}$ is also extensional w.r.t. E, which implies $\mathcal{A} \subseteq \mathcal{B}$. Since $\mathcal{B}$ satisfies the closure properties required for $\mathcal{A}^*$, we have $\mathcal{A}^* \subseteq \mathcal{B}$, and therefore

$$E_{\mathcal{A}^*} \;\geq\; E_{\mathcal{B}} \;=\; E_{\mathcal{B}^*} \;=\; E.$$

In order to prove $E_{\mathcal{A}^*} \leq E$, we need the laws

- $(\alpha \wedge \beta)^k \;\leq\; \alpha^k \wedge \beta^k$, which is easily proved by induction, and
- $(\alpha \rightarrow \beta)^k \;\leq\; \alpha^k \rightarrow \beta^k$. This formula is derived by residuation from

$$\beta^k \;\geq\; (\alpha \wedge \beta)^k \;=\; (\alpha * (\alpha \rightarrow \beta))^k \;=\; \alpha^k * (\alpha \rightarrow \beta)^k.$$

These two laws imply $(\alpha \leftrightarrow \beta)^k \;\leq\; \alpha^k \leftrightarrow \beta^k$. Thus, taking (4) into account, we have for all $\mu \in \mathcal{A}$

$$(\mu(x) \leftrightarrow \mu(y))^k \;\geq\; \mu(x)^k \leftrightarrow \mu(y)^k \;\geq\; E_{\mathcal{A}^*},$$

since $\mu^k \in \mathcal{A}^*$ and therefore μ^k has to be extensional w.r.t. $E_{\mathcal{A}^*}$. This proves also $E_{\mathcal{A}^*} \leq E$. $\square$

We only considered unary predicates that are associated with fuzzy sets on X. If we consider also n-ary predicates, we only need to consider the additional unary predicates that we obtain by instantiating all variables but one of the n-ary predicates. In this way, we can again compute the corresponding equality relation $E_{\mathcal{A}^*}$ on the basis of the enriched set $\mathcal{A}$. Obviously, the fuzzy sets $\mu(x_1, \ldots, x_n)$ associated with the n-ary predicates are then extensional w.r.t. the equality relation

$$E((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \;=\; E_{\mathcal{A}^*}(x_1, y_1) * \ldots * E_{\mathcal{A}^*}(x_n, y_n). \qquad (13)$$

But since the values $E_{\mathcal{A}^*}(x_i, y_i)$ are idempotent, we may replace $*$ in (13) by $\wedge$.

Another interesting remark is that for $* = \wedge$, we have $E_{\mathcal{A}^*} = E_{\mathcal{A}}$, i.e., it is sufficient to consider only the elementary predicates in this case.

Usually, fuzzy logic in the narrow sense with $L = [0, 1]$ as the underlying lattice is based on the Łukasiewicz implication $\alpha \to \beta = \min\{1 - \alpha + \beta, 1\}$, meaning that $*$ is the Łukasiewicz conjunction $\alpha * \beta = \max\{\alpha + \beta - 1, 0\}$. The reason for this is that for soundness and completeness the implication has to be continuous and the Łukasiewicz implication is – up to isomorphism – the only continuous residuated implication [14]. Since the Łukasiewicz conjunction is nilpotent, this means that the equality relation $E_{\mathcal{A}^*}$ coincides with the crisp equality – at least if for all $x, y \in X$, $x \neq y$, there exists a fuzzy set $\mu \in \mathcal{A}$ (elementary predicate) such that $\mu(x) \neq \mu(y)$ holds. This means that this logic still maintains the potential for distinguishing objects well. Also when the product is chosen as the underlying t-norm the corresponding logic has the potential for distinguishing objects well. More generally, this applies to all t-norms whose only idempotent elements are 0 and 1.

References

[1] A. Di Nola, S. Sessa, W. Pedrycz, E. Sanchez, Fuzzy Relation Equations and their Applications to Knowledge Engineering. Kluwer Academic Publishers, Dordrecht (1989).

[2] D. Dubois, H. Prade, Similarity–Based Approximate Reasoning. In: J.M. Zurada, R.J. Marks II, C.J. Robinson (eds.), Computational Intelligence Imitating Life. IEEE Press, New York (1994), 69-80.

[3] S. Gottwald, Fuzzy Sets and Fuzzy Logic. Vieweg, Wiesbaden (1993).

[4] U. Höhle, M–valued Sets and Sheaves over Integral Commutative CL–Monoids. In: S.E. Rodabaugh, E.P. Klement, U. Höhle (eds.), Applications of Category Theory to Fuzzy Subsets. Kluwer Academic Publishers, Dordrecht (1992), 33-72.

[5] U. Höhle, Monoidal Logics. In: R. Kruse, J. Gebhardt, R. Palm (eds.), Fuzzy Systems in Computer Science. Vieweg, Braunschweig (1994), 233-243.

[6] U. Höhle, L.N. Stout, Foundations of Fuzzy Sets. Fuzzy Sets and Systems 40 (1991), 257-296.

[7] J. Jacas, J. Recasens, Eigenvectors and Generators of Fuzzy Relations. Proc. IEEE International Conference on Fuzzy Systems 1992, San Diego (1992), 687–694.

[8] F. Klawonn, J.L. Castro: Similarity in Fuzzy Reasoning. Mathware and Soft Computing 2 (1995), 336-350.

[9] F. Klawonn, R. Kruse, Equality Relations as a Basis for Fuzzy Control. Fuzzy Sets and Systems 54 (1993), 147-156.

[10] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic. Prentice Hall, Upper Saddle River, NJ (1995).

[11] R. Kruse, J. Gebhardt, F. Klawonn, Foundations of Fuzzy Systems. Wiley, Chichester (1994).

[12] V. Novák, Fuzzy Sets and their Applications. Adam-Hilger, Bristol (1989).

[13] V. Novák, On the Syntactico-Semantical Completeness of First-Order Fuzzy Logic. Part I – Syntactical Aspects; Part II – Main Results. Kybernetika 26 (1990), 47-66; 134-154.

[14] J. Pavelka, On Fuzzy Logic I, II, III. Zeit. Math. Logik Grundl. Math. 25 (1979), 45-52, 119-134, 447-464.

[15] E.H. Ruspini, On the Semantics of Fuzzy Logic. Intern. Journ. of Approximate Reasoning 5 (1991), 45-88.

[16] N. Schmechel, On the Lattice-Isomorphism Between Fuzzy Equivalence Relations and Fuzzy Partitions. In: 25th Intern. Symposium on Multi-Valued Logic. Bloomington, Indiana (1995), 146-151.

[17] H. Thiele, On the Mutual Definability of Fuzzy Toelrance Relations and Fuzzy Tolerance Coverings. In: 25th Intern. Symposium on Multi-Valued Logic. Bloomington, Indiana (1995), 140-145.

[18] H. Thiele, A Characterization of Ruspini-Partitions by Similarity Relations. In: 6th Intern. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems Vol.I. Granada (1996), 389-394.

[19] H. Thiele, On Isomorphisms Between the Lattice of Tolerance Relations and Lattices of Clusterings. In: 26th Intern. Symposium on Multi-Valued Logic. Santiago de Compostela (1996), 198-202.

[20] H. Thiele, On Isomorphisms Between the DeMorgan Algebra of Fuzzy Tolerance Relations and DeMorgan Algebras of Fuzzy Clusterings. In: 5th IEEE Intern. Conf. on Fuzzy Systems, Vol. II. New Orleans (1996), 960-965.

[21] H. Thiele, N. Schmechel, The Mutual Defineability of Fuzzy Equivalence Relations and Fuzzy Partitions. Proc. Intern. Joint Conference of the Fourth IEEE International Conference on Fuzzy Systems and the Second International Fuzzy Engineering Symposium, Yokohama (1995), 1383-1390.

[22] E. Trillas, L. Valverde, An Inquiry into Indistinguishability Operators. In: H.J. Skala, S. Termini, E. Trillas (eds.), Aspects of Vagueness. Reidel, Dordrecht (1984), 231-256.

[23] E. Trillas, L. Valverde, On Implication and Indistinguishability in the Setting of Fuzzy Logic. In: J. Kacprzyk, R. Yager (eds.), Management Decision Support Systems Using Fuzzy Sets and Possibility Theory. Verlag TÜV Rheinland, Köln (1985), 198-212.

[24] L. Valverde, On the Structure of F–Indistinguishability Operators. Fuzzy Sets and Systems 17 (1985), 313-328.

[25] L.A. Zadeh, Similarity Relations and Fuzzy Orderings. Information Sciences 3 (1971), 177-200.