Fuzzy Points, Fuzzy Relations and Fuzzy Functions

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Abstract

In many applications of fuzzy sets, especially in fuzzy control, the notions of fuzzy points and fuzzy functions play an important role. Nevertheless, these concepts are usually understood and used on a very intuitive basis without providing a formal definition. In this paper we discuss an approach to clarify and formalize these notions and show some consequences for fuzzy control and fuzzy interpolation. One of the main results shows that the Mamdani fuzzy controller provides a solution to the system of fuzzy relational equations induced by the if-then-rules if it determines a partial fuzzy function.

1 Introduction

When L.A. Zadeh introduced the notion of a fuzzy set in his seminal paper [24] in 1965 he was motivated by the idea to model vague (linguistic) concepts like *small*, *long*, etc. The generalization from crisp membership of an element to a set to a gradual membership requires to transfer definitions and operations for crisp sets to fuzzy sets. This is often done by formalizing the concept to be generalized for ordinary sets in classical logic and then extending this definition to a $[0,1]$-valued (fuzzy) logic. An example of this approach is Zadeh's extension principle [26], that proposes a method to extend a function defined on a crisp domain to fuzzy sets. The extension principle is a good example for a general problem in fuzzy set theory that often appears when crisp concepts are generalized. Taking a closer look at the extension principle it consists of two steps. Starting with an ordinary function $f : X \longrightarrow Y$ (defined on points) the first step extends f to the power sets of X and Y in the usual way by defining $f[M] = \{f(x) \mid x \in M\}$ for a subset $M \subseteq X$. Then this definition of a function on *sets* is extended to fuzzy sets in the second step. This means that the extension to fuzzy sets does not preserve the (fuzzified) properties of a function, but properties of (fuzzy) set-valued set function.

A good example for the problems connected with this approach is fuzzy arithmetics, where operations like the sum or the product of real numbers are extended to fuzzy numbers or fuzzy sets (for details see for instance [16]). Fundamental laws like distributivity or the group axioms for + no longer hold in fuzzy arithmetics. The reason for this is not the extension to fuzzy sets but the first inherent step where the arithmetic operations are extended to sets of real numbers. Already in this first step many of the algebraic properties are lost.

The crucial point is that when dealing with sets the concept of a singleton or point – a set with just one element – does no longer play an important role and is often not considered separately when operating with sets, although it is not a formal or principal problem. Thus a suitable definition of the concept of a fuzzy point is needed in order to distinguish between general fuzzy sets and fuzzy singletons.

A very naive approach to fuzzy points simply defines a fuzzy point as a fuzzy set with membership degree 1 for exactly one element and zero membership for all other elements. However, such fuzzy sets are not typical for applications. Furthermore, the idea of a fuzzy point should in some way be an appropriate model for a vague value like *approximately zero*. The fundamental concept on which the concept of a fuzzy point is based in this paper is the notion indistinguishability or fuzzy equality. Starting with notion a fuzzy point corresponds to the fuzzy set of all elements that are indistinguishable from (or fuzzy equal to) a given crisp point. Therefore we first introduce fuzzy equality in Section 2 in order to discuss the concept of a fuzzy point in Section 3. Section 4 is devoted to fuzzy functions and establishes the close connections to fuzzy control and fuzzy interpolation.

2 GL-Monoids and Fuzzy Equality

In the setting of fuzzy systems for our considerations we essentially need a continuous t-norm together with induced operations like residuated implication and biimplication. The fundamental formal properties we need do not come from the rich structure of the unit interval, but mainly from the concept of residuation. In order to concentrate on the essential structure we choose GL-monoids as the more general formal framework for our investigations instead of the unit interval. From [5], we recall the definition of a GL-monoid.

Definition 1 $(L, \leq, *)$ *is a GL-monoid if*

1. $(L, \leq)$ *is a complete lattice,*

2. $(L, *, \mathbb{1}, \mathbb{0})$ *is a commutative monoid with unit* $\mathbb{1}$ *and zero element* $\mathbb{0}$*, i.e. the operation* $* : L \times L \longrightarrow L$ *is associative and commutative and the equations* $\mathbb{1} * \alpha = \alpha$ *and* $\alpha * \mathbb{0} = \mathbb{0}$ *hold for* $\alpha \in L$*,*

3. $*$ *is isotone, i.e.*
$$\alpha \leq \beta \implies \alpha * \gamma \leq \beta * \gamma,$$

4. $(L, \leq, *)$ *is integral, i.e.* $\mathbb{1} = \bigvee L$

5. $(L, \leq, *)$ *is the dual of a divisibility monoid, i.e.*
$$\alpha \leq \beta \text{ implies the existence of } \gamma \in L \text{ such that } \alpha = \beta * \gamma,$$

6. $(L, \leq, *)$ *is residuated, meaning that there exists a binary operation* $\rightarrow$ *on* L *satisfying*
$$\alpha * \beta \leq \gamma \iff \alpha \leq \beta \rightarrow \gamma, \tag{1}$$

7. *the infinite distributive law holds, i.e.*
$$\alpha * \bigvee_{i \in I} \beta_i \;=\; \bigvee_{i \in I} (\alpha * \beta_i).$$

For some basic properties of GL-monoids we refer to [5, 12]. Unless otherwise stated we assume $(L, \leq, *)$ to be a GL-monoid. $\vee$ can be considered as the valuation function of a disjunction, $\wedge$ and $*$ as two alternatives for a conjunction in an L-valued logic. The binary operation $\rightarrow$ is uniquely determined by the adjunction property (1):
$$\alpha \rightarrow \beta \;=\; \bigvee \{\lambda \in L \mid \alpha * \lambda \leq \beta\}. \tag{2}$$
$\rightarrow$ can be viewed as the valuation function for the implication (associated with the conjunction $*$). In a straight forward manner we define the biimplication
$$\alpha \leftrightarrow \beta \;=\; (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha).$$

For a more detailed discussion of GL-monoids and their properties we refer to [5, 12] and [6], the latter one being devoted to the relation between logical calculi and structures like GL-monoids.

In the framework of fuzzy sets L is the unit interval with the usual linear ordering. $*$ can be any continuous t-norm (a commutative, associative, non-decreasing binary operation on $[0, 1]$ having 1 as unit).

Example 1 Let $L = [0, 1]$ be the unit interval with the usual ordering. Then
$$\alpha \leftrightarrow \beta \;=\; \max\{\alpha, \beta\} \rightarrow \min\{\alpha, \beta\}$$
holds [22]. It is easy to check that based on the choice of $*$ the following formula can be derived for $\rightarrow$ and $\leftrightarrow$ (cf. [9, 16]).

$\alpha * \beta$	$\max\{\alpha+\beta-1, 0\}$	$\min\{\alpha, \beta\}$	$\alpha \cdot \beta$
$\alpha \to \beta$	$\min\{1-\alpha+\beta, 1\}$	$\begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha \leq \beta \\ \frac{\beta}{\alpha} & \text{otherwise} \end{cases}$
$\alpha \leftrightarrow \beta$	$1 - \lvert\alpha - \beta\rvert$	$\begin{cases} 1 & \text{if } \alpha = \beta \\ \min\{\alpha, \beta\} & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha = \beta \\ \frac{\min\{\alpha,\beta\}}{\max\{\alpha,\beta\}} & \text{otherwise} \end{cases}$

An L-fuzzy (sub)set (or simply a fuzzy set) of the set X is a mapping $\mu : X \longrightarrow L$, assigning to element $x \in X$ its membership degree $\mu(x)$ to μ.

Definition 2 *An equality relation (w.r.t. the operation $*$) on the set X is a mapping $E : X \times X \longrightarrow L$ satisfying the axioms:*

(E1)	$E(x, x) = 1,$	(reflexivity)
(E2)	$E(x, y) = E(y, x),$	(symmetry)
(E3)	$E(x, y) * E(y, z) \leq E(x, z).$	(transitivity)

When L is the unit interval and depending on the choice of the operation $*$, sometimes E is also called a similarity relation [25, 17], indistinguishability operator [21], fuzzy equality (relation) [8, 13], fuzzy equivalence relation [20] or proximity relation [2].

Interpreting indistinguishability as the dual concept to distance, a (pseudo-)metric δ on a set X induces an equality relation w.r.t. the Łukasiewicz t-norm $\alpha * \beta = \max\{\alpha + \beta - 1, 0\}$ ($L = [0, 1]$) by $E(x, y) = 1 - \min\{\delta(x, y), 1\}$. Vice versa, an equality relation E w.r.t. the Łuksaiewicz t-norm induces a (pseudo-)metric by $\delta(x, y) = 1 - E(x, y)$.

An equality relation E is intended to model gradual indistinguishability between elements. For the usual crisp equality we can substitute equal elements in any formula or predicate without effecting the truth value of the formula. In the context of fuzzy sets we call this substitution property extensionality.

Definition 3 *A fuzzy set $\mu \in L^X$ is called extensional w.r.t. the equality relation E on X if*

$$\mu(x) * E(x, y) \;\leq\; \mu(y)$$

holds for all $x, y \in X$.

The fuzzy set

$$\hat{\mu} \;=\; \bigwedge\{\nu \mid \mu \leq \nu \text{ and } \nu \text{ is extensional w.r.t. } E\}$$

is called the extensional hull of μ w.r.t. E.

The extensional hull of a fuzzy set is the smallest extensional fuzzy set containing this fuzzy set. Taking the residuation property into account we can directly derive from the definition of extensionality that a fuzzy set $\mu \in L^X$ is extensional w.r.t. the equality relation E if and only if

$$\mu(x) \leftrightarrow \mu(y) \;\geq\; E(x,y) \tag{3}$$

holds for all $x, y \in X$.

For any fuzzy set μ we can define an equality relation $E_\mu(x,y) = \mu(x) \leftrightarrow \mu(y)$, having the property that it is the coarsest equality relation for which μ is extensional. Theorem 1 in the following section will provide a more general result.

The extensional hull of a crisp set M or an element x_0 is defined in the canonical way by building the extensional hulls of the characteristic functions of M and $\{x_0\}$, respectively. Thus the extensional hull of M is given by

$$\hat{M}(x) \;=\; \bigvee_{m \in M} E(x,m),$$

whereas the extensional hull of x_0 is simple

$$\hat{x_0}(x) \;=\; E(x, x_0).$$

We obtain an interesting example of an extensional hull of point x_0 when we consider the equality relation on the real numbers induced by the standard metric on $\mathbb{R}$, i.e.

$$E(x,y) \;=\; 1 - \min\{|x-y|, 1\}.$$

In this case, the extensional hull of the point x_0 is the fuzzy set with the triangular membership function, having a membership degree of one at x_0 and a membership degree of 0 at $x_0 - 1$ and $x_0 + 1$. Other triangular membership functions and even other shapes as extensional hulls are obtained when a scaling of the metric or problem dependent (monotonous) transformation is applied to the real line [11].

When we consider a singleton as a set with exactly one element, i.e. at least and at most one element, we can easily generalize this definition in the context of fuzzy sets.

Definition 4 *A fuzzy set $\mu \in L^X$ is called a singleton w.r.t. the equality relation E on X if*

(i) $\mu(x) * \mu(y) \leq E(x,y)$

(ii) $\bigvee_{x \in X} \mu(x) = \mathbb{1}$

hold.

(i) reflects that when x and y belong both to the singleton μ then x and y must be equal. (ii) ensures that μ is not empty.

It is easy to check that the extensional hull of an element is a singleton. However, it is not necessary that a singleton corresponds to the extensional hull of an element.

Example 2 Let $X = (0,1)$, $L = [0,1]$ and let $*$ be the Łukasiewicz t-norm. We consider the equality relation E on X induced by the standard metric. The fuzzy set $\mu(x) = x$ is indeed a singleton, but it is not the extensional hull of any element of X, since 1 does not belong to X.

A similar notion of a singleton was already introduced in [4].

3 Fuzzy Points

Before we take a closer look at singletons have to establish another connection between fuzzy sets and equality relations. In the previous section we have introduced the extensional hull of fuzzy set w.r.t. to an equality relation. Now we construct an equality relation for a given set of fuzzy sets such that all these fuzzy sets are extensional.

Theorem 1 *Let $\mathcal{F} \subseteq L^X$ be a set of fuzzy sets. Then*

$$E_{\mathcal{F}}(x,y) \;=\; \bigwedge_{\mu\in\mathcal{F}} (\mu(x) \leftrightarrow \mu(y)) \tag{4}$$

is the coarsest (greatest) equality relation on X such that all fuzzy sets in $\mathcal{F}$ are extensional w.r.t. $E_{\mathcal{F}}$.

Proof. It is obvious that $E_{\mathcal{F}}$ is reflexive and symmetric. The transitivity of $E_{\mathcal{F}}$ follows from

$$\begin{aligned}
E_{\mathcal{F}}(x,y) * E_{\mathcal{F}}(y,z) &= \left(\bigwedge_{\mu\in\mathcal{F}} (\mu(x) \leftrightarrow \mu(y))\right) * \left(\bigwedge_{\nu\in\mathcal{F}} (\nu(y) \leftrightarrow \nu(z))\right) \\
&\leq \bigwedge_{\mu,\nu\in\mathcal{F}} \Big((\mu(x) \leftrightarrow \mu(y)) * (\nu(y) \leftrightarrow \nu(z))\Big) \\
&\leq \bigwedge_{\mu\in\mathcal{F}} \Big((\mu(x) \leftrightarrow \mu(y)) * (\mu(y) \leftrightarrow \mu(z))\Big) \\
&\leq \bigwedge_{\mu\in\mathcal{F}} (\mu(x) \leftrightarrow \mu(z)) \\
&= E_{\mathcal{F}}(x,z)
\end{aligned}$$

where we made use of the facts

$$\alpha * \bigwedge_{i\in I} \beta_i \;\leq\; \bigwedge_{i\in I} (\alpha * \beta_i)$$

and

$$(\alpha \leftrightarrow \beta) * (\beta \leftrightarrow \gamma) \;\leq\; \alpha \leftrightarrow \gamma$$

that are valid in any GL-monoid (see for instance [12]).

The extensionality of the fuzzy sets in $\mathcal{F}$ follows directly from Equation (3) and the definition of $E_{\mathcal{F}}$.

Finally, we have to show that $E_{\mathcal{F}}$ is the coarsest equality relation making all fuzzy sets in $\mathcal{F}$ extensional. Let E be an equality relation such that all fuzzy sets in $\mathcal{F}$ are extensional w.r.t. E. By Equation (3), $E(x,y) \leq \mu(x) \leftrightarrow \mu(y)$ holds for all $\mu \in \mathcal{F}$ which implies $E(x,y) \leq E_{\mathcal{F}}(x,y)$. □

In the setting of $L = [0,1]$ the equality relation (4) already appeared in Valverde's representation theorem [23].

We now return to our considerations on fuzzy points. In the previous section we have seen an interesting aspect of equality relations. A crisp set induces a fuzzy set as its extensional hull and in the very special case of Example 2 the fuzzy set induced by a point has a triangular membership function. Interpreting a fuzzy set as the extensional hull of a single point, i.e., as a kind of vague value, will provide a deeper insight to a variety of fuzzy systems, especially fuzzy controllers. Therefore, in the following we investigate the question when a given set of fuzzy sets can be interpreted as 'fuzzy points', i.e., as extensional hulls of crisp points.

Theorem 2 *Let $(\mu_i)_{i\in I} \subseteq L^X$ be a family of fuzzy sets and let $(x_i)_{i\in I} \subseteq X$ be a family of elements of X such that $\mu_i(x_i) = \mathbb{1}$ holds for all $i \in I$. Then the following two statements are equivalent.*

(i) There exists an equality relation on X such that the extensional hull of x_i with respect to E equals μ_i for all $i \in I$, i.e.

$$\mu_i(x) \;=\; E(x, x_i). \tag{5}$$

(ii) For all $i, j \in I$

$$\bigvee_{x\in X} (\mu_i(x) * \mu_j(x)) \;\leq\; \bigwedge_{y\in X} (\mu_i(y) \leftrightarrow \mu_j(y)) \tag{6}$$

holds.

Proof. We first prove that (i) implies (ii). We have to show that for any $x, y \in X$ the inequality

$$\mu_i(x) * \mu_j(x) \;\leq\; \mu_i(y) \leftrightarrow \mu_j(y) \tag{7}$$

holds. According to (i) there exists an equality relation E such that (5) holds so that we can rewrite (7) as

$$\begin{aligned} E(x,x_i) * E(x,x_j) \;&\leq\; E(y,x_i) \leftrightarrow E(y,x_j) \\ &=\; (E(y,x_i) \to E(y,x_j)) \wedge (E(y,x_j) \to E(y,x_j)). \end{aligned}$$

For reasons of symmetry it is sufficient to prove

$$E(x, x_i) * E(x, x_j) \ \leq \ E(y, x_i) \to E(y, x_j). \tag{8}$$

Using the adjunction property (8) is equivalent to

$$E(x, x_i) * E(x, x_j) * E(y, x_i) \ \leq \ E(y, x_j)$$

which is satisfied because of the transitivity of E so that we have proved (i) implies (ii).

To show that also the other implication (ii)$\Rightarrow$(i) holds, let

$$E(x, y) \ = \ \bigwedge_{i \in I} (\mu_i(x) \leftrightarrow \mu_i(y)) \,. \tag{9}$$

Theorem 1 guarantees that E is an equality relation. Furthermore, from the fact that $\alpha \leftrightarrow \mathbb{1} = \alpha$ holds in any GL-monoid we derive

$$\begin{aligned} E(x, x_i) \ &\leq \ \mu_i(x) \leftrightarrow \mu_i(x_i) \\ &= \ \mu_i(x) \leftrightarrow \mathbb{1} \\ &= \ \mu_i(x). \end{aligned}$$

What remains to be proved is $\mu_i(x) \leq E(x, x_i)$. For this we have to show that

$$\mu_i(x) \ \leq \ \mu_j(x) \leftrightarrow \mu_j(x_i) \ = \ (\mu_j(x) \to \mu_j(x_i)) \wedge (\mu_j(x_i) \to \mu_j(x))$$

holds for all $j \in I$, or equivalently that

$$\mu_i(x) \ \leq \ \mu_j(x) \to \mu_j(x_i) \tag{10}$$

and

$$\mu_i(x) \ \leq \ \mu_j(x_i) \to \mu_j(x) \tag{11}$$

are satisfied.

Using the adjunction property for (10) we have to prove

$$\mu_i(x) * \mu_j(x) \ \leq \ \mu_j(x_i). \tag{12}$$

Because of the assumption (6) we have

$$\begin{aligned} \mu_i(x) * \mu_j(x) \ &\leq \ \bigvee_{z \in X} (\mu_i(z) * \mu_j(z)) \\ &\leq \ \bigwedge_{y \in X} (\mu_i(y) \leftrightarrow \mu_j(y)) \\ &\leq \ \mu_i(x_i) \leftrightarrow \mu_j(x_i) \\ &= \ \mathbb{1} \leftrightarrow \mu_j(x_i) \\ &= \ \mu_j(x_i) \end{aligned}$$

so that (10) is proved.

Using the adjunction property for (11) we need to prove

$$\mu_i(x) * \mu_j(x_i) \ \leq \ \mu_j(x),$$

or equivalently, again by adjunction

$$\mu_j(x_i) \ \leq \ \mu_i(x) \to \mu_j(x). \tag{13}$$

Taking the assumption (6) into account we obtain

$$\begin{aligned}
\mu_i(x) \to \mu_j(x) \ &\geq \ \mu_i(x) \leftrightarrow \mu_j(x) \\
&\geq \ \bigwedge_{y \in X} (\mu_i(y) \leftrightarrow \mu_j(y)) \\
&\geq \ \bigvee_{z \in X} (\mu_i(z) * \mu_j(z)) \\
&\geq \ \mu_i(x_i) * \mu_j(x_i) \\
&= \ \mathbb{1} * \mu_j(x_i) \\
&= \ \mu_j(x_i).
\end{aligned}$$

Thus (13) also holds. □

There are some interesting remarks to Theorem 2.

(a) The necessary and sufficient condition (6) can be interpreted in the sense that the degree of non-disjointness of the fuzzy sets μ_i and μ_j must not exceed their degree of equality/equivalence. In the crisp case this is the usual requirement that two equivalence classes are either equal or disjoint.

(b) The necessary and sufficient condition (6) can be weakened to a sufficient one by requiring that the fuzzy sets are disjoint w.r.t. $*$, i.e., for all $i \neq j$ we have $\mu_i(x) * \mu_j(x) = \mathbf{0}$ for all x.

(c) The proof is constructive in the sense that in case condition (6) is satisfied, a corresponding equality relation is explicitly given by formula (9).

(d) Condition (6) was only needed for the last part of the proof showing that $\mu_i(x) \leq E(x, x_i)$ holds. Thus the equality relation (9) always fulfills $E(x, x_i) \leq \mu_i(x)$.

(e) A proof of this theorem for $L = [0,1]$ and $*$ a continuous t-norm was already given in [16]. The proof provided in the more general framework of GL-monoids is significantly shorter and simpler than the one for the unit interval.

(f) The first formulation of this theorem appeared in [7].

Corollary 1 *Let $(\mu_i)_{i\in I} \subseteq L^X$ be a family of fuzzy sets and let $(x_i)_{i\in I} \subseteq X$ be a family of elements of X such that $\mu_i(x_i) = \mathbb{1}$ holds for all $i \in I$. If condition (6) of Theorem 2 holds, then the equality relation (9) is the coarsest equality relation satisfying condition (5) of Theorem 2.*

Proof. According to Equation (3) and the definition (9) of the equality relation E, the fuzzy sets μ_i are extensional with respect to E. For any equality relation $\tilde{E}$ satisfying condition (5) the fuzzy sets μ_i are extensional with respect to $\tilde{E}$ because of the transitivity of $\tilde{E}$:

$$\mu_i(x) * \tilde{E}(x,y) \;=\; \tilde{E}(x,x_i) * \tilde{E}(x,y) \;\leq\; \tilde{E}(y,x_i) \;=\; \mu_i(y)$$

But due to Theorem 1 E is the coarsest equality relation making the fuzzy sets μ_i extensional. $\square$

Theorem 3 *Let $(\mu_i)_{i\in I} \subseteq L^X$ be a family of fuzzy sets and let $(x_i)_{i\in I} \subseteq X$ be a family of elements of X such that $\mu_i(x_i) = \mathbb{1}$ holds for all $i \in I$. If condition (6) of Theorem 2 holds, then the equality relation*

$$E(x,y) \;=\; \begin{cases} 1 & \text{if } x = y \\ \bigvee_{i\in I} (\mu_i(x) * \mu_i(y)) & \text{otherwise} \end{cases} \tag{14}$$

is the smallest (finest) equality relation satisfying condition (5) of Theorem 2.

Proof. Obviously, $E(x,x) = 1$ holds and E is symmetric. To prove the transitivity of E we use condition (6).

$$\begin{aligned} E(x,y) * E(y,z) &= \bigvee_{i\in I}\bigvee_{j\in I} (\mu_i(x) * \mu_i(y) * \mu_j(y) * \mu_j(z)) \\ &\leq \bigvee_{i\in I}\bigvee_{j\in I} \left(\mu_i(x) * \left(\bigvee_{s\in X} (\mu_i(s) * \mu_j(s))\right) * \mu_j(z)\right) \\ &\leq \bigvee_{i\in I}\bigvee_{j\in I} \left(\mu_i(x) * \left(\bigwedge_{s\in X} (\mu_i(s) \leftrightarrow \mu_j(s))\right) * \mu_j(z)\right) \end{aligned}$$

$$
\begin{aligned}
&\leq \bigvee_{i\in I}\bigvee_{j\in I}\Big(\mu_i(x) * (\mu_i(x) \leftrightarrow \mu_j(x)) * \mu_j(z)\Big) \\
&\leq \bigvee_{i\in I}\bigvee_{j\in I}\Big(\mu_i(x) * (\mu_i(x) \rightarrow \mu_j(x)) * \mu_j(z)\Big) \\
&\leq \bigvee_{i\in I}\bigvee_{j\in I}(\mu_j(x) * \mu_j(z)) \\
&= \bigvee_{j\in I}(\mu_j(x) * \mu_j(z)) \\
&= E(x,z)
\end{aligned}
$$

which shows that E is transitive.

Since $\mu_j(x_j) = \mathbb{1}$, we have

$$E(x,x_j) \;=\; \bigvee_{i\in I}(\mu_i(x_j) * \mu_i(x)) \;\geq\; \mu_j(x_j) * \mu_j(x) \;=\; \mu_j(x). \tag{15}$$

Using again $\mu_j(x_j) = \mathbb{1}$ and (6), we obtain

$$
\begin{aligned}
E(x,x_j) &= \mu_j(x_j) * E(x,x_j) \\
&= \bigvee_{i\in I}(\mu_j(x_j) * \mu_i(x_j) * \mu_i(x)) \\
&\leq \bigvee_{i\in I}\left(\left(\bigvee_{y\in X}(\mu_j(y) * \mu_i(y))\right) * \mu_i(x)\right) \\
&\leq \bigvee_{i\in I}\left(\left(\bigwedge_{y\in X}(\mu_j(y) \leftrightarrow \mu_i(y))\right) * \mu_i(x)\right) \\
&\leq \bigvee_{i\in I}((\mu_j(x) \leftrightarrow \mu_i(x)) * \mu_i(x)) \\
&\leq \bigvee_{i\in I}((\mu_i(x) \rightarrow \mu_j(x)) * \mu_i(x)) \\
&= \bigvee_{i\in I}(\mu_i(x) \wedge \mu_j(x)) \\
&\leq \mu_j(x)
\end{aligned}
$$

so that together with (15) $E(x,x_j) = \mu_j(x)$ is proved.

Finally, let $\tilde{E}$ be an equality relation also satisfying $\tilde{E}(x,x_j) \;=\; \mu_j(x)$. Because

$$\tilde{E}(x,y) \;\geq\; \tilde{E}(x,x_i) * \tilde{E}(x_i,y) \;=\; \mu_i(x) * \mu_i(y)$$

holds for all $i \in I$, we conclude $E(x,y) \leq \tilde{E}(x,y)$ for all $x, y \in X$. $\square$

Corollary 2 *Let $L = [0,1]$ and $* = \wedge$. Let $(\mu_i)_{i \in I} \subseteq L^X$ be a family of fuzzy sets and let $(x_i)_{i \in I} \subseteq X$ be a family of elements of X such that $\mu_i(x_i) = \mathbb{1}$ holds for all $i \in I$. If for each $x \in X$ there exists a pair $(i,j) \in I \times I$ such that $\mu_i(x) \neq \mu_j(x)$, then there is at most one equality relation satisfying condition (5) of Theorem 2.*

Proof. According to Theorem 2 there exists an equality relation satisfying condition (5) only if (6) holds. In this case, the equality relation (14) is the smallest one satisfying condition (5) (see Theorem 3). Due to Corollary 1 we also know the greatest equality relation that has property (5), namely the one given in (9).

Recalling Example 1, we have for $L = [0,1]$ and $* = \wedge$

$$\alpha \leftrightarrow \beta \; = \; \begin{cases} 1 & \text{if } \alpha = \beta \\ \alpha \wedge \beta & \text{otherwise.} \end{cases}$$

Taking this fact into account as well as the assumption that for each $x \in X$ there exists a pair $(i,j) \in I \times I$ such that $\mu_i(x) \neq \mu_j(x)$, we can rewrite the greatest equality relation (9) satisfying condition (5) in the form

$$E(x,y) \; = \; \begin{cases} 1 & \text{if } x = y \\ \bigwedge\limits_{i \in I} (\mu_i(x) \wedge \mu_i(y)) & \text{otherwise} \end{cases}$$

which is smaller than or equal to the smallest solution (14). Thus, the smallest and greatest solution coincide. $\square$

The equality relation (14) was already defined in [18, 20] where a version of Theorem 2 for arbitrary t-norms is proved, however, assuming in addition that the considered fuzzy sets cover the underlying set X to the degree 1. In this way, a one-to-one correspondence between equality relations and fuzzy partitions as they are defined in [20] can be established. In [19] weaker notions of fuzzy partitions are considered and the corresponding structures replacing equality relations for establishing a one-to-one correspondence with these weaker fuzzy partitions are investigated. Since (fuzzy) partitions are not our main concern, we do not dive into a deeper discussion of this topic.

4 Fuzzy Functions

Now that we have clarified our understanding of a fuzzy point, we start our investigations on fuzzy functions. But before we start with the formal considerations we take brief look at fuzzy relations in approximate reasoning, especially in fuzzy control.

In approximate reasoning if-then rules of the form

$$\text{If } \xi \text{ is } A, \text{ then } \eta \text{ is } B, \tag{16}$$

are very common where ξ and η are variables with domains X and Y, respectively. A and B are linguistic terms like *positive big* or *approximately zero* (see, e.g., [15]). These linguistic terms are usually modelled by suitable fuzzy sets, say $\mu_A \in L^X$ and $\mu_B \in L^Y$. In addition to such general rules one has specific information like

$$\xi \text{ is } A' \tag{17}$$

where A' is represented by the fuzzy set $\mu_{A'} \in L^X$ (or simply by $\mu \in L^X$).

The application of a single rule of the form (16) to the information (17) is usually formalized on the basis of a computing scheme of the following form. The rule is encoded as a fuzzy relation of the form

$$\varrho(x,y) \;=\; \varrho_{\odot}(x,y) \;=\; \mu_A(x) \odot \mu_B(y) \tag{18}$$

where $\odot$ is either the operation $*$ or the residuated implication $\rightarrow$. For a given input information in the form of the fuzzy set $\mu_{A'} \in L^X$, the 'output' fuzzy set $\nu_{\text{conclusion}}$ is computed as the composition of the fuzzy relation $\varrho_{\odot}$ and the fuzzy set $\mu_{A'}$, i.e.

$$\varrho[\mu_{A'}](y) \;=\; \bigvee_{x \in X} \{\mu_{A'}(x) * \varrho(x,y)\} \tag{19}$$

for all $y \in Y$ (cf., e.g., [1, 3, 15]). This scheme is called sup-$*$-inference. In fuzzy control, for instance, usually $* = \min = \odot$ is chosen.

For a collection of if-then rules of the form

$$\text{If } \xi \text{ is } A_i, \text{ then } \eta \text{ is } B_i, \qquad (i \in I), \tag{20}$$

where the linguistic terms A_i and B_i are modelled by the fuzzy set $\mu_{A_i} \in L^X$ and $\mu_{B_i} \in L^Y$ the output fuzzy set for a given 'input fuzzy set' $\mu \in L^X$ is usually computed either by

$$\bigwedge_{i \in I} \varrho_i[\mu], \tag{21}$$

when the fuzzy relations ϱ_i are of the type $\varrho_{\rightarrow}$, or by

$$\bigvee_{i \in I} \varrho_i[\mu], \tag{22}$$

when the fuzzy relations ϱ_i are of the type ϱ_*. In other words, we associate with the collection (20) of if-then-rules either the fuzzy relation

$$\varrho_U(x,y) \;=\; \bigwedge_{i \in I} \varrho_{\rightarrow}(x,y) \;=\; \bigwedge_{i \in I} (\mu_i(x) \rightarrow \nu_i(y)) \tag{23}$$

or the fuzzy relation

$$\varrho_L(x,y) \;=\; \bigvee_{i\in I} \varrho_*(x,y) \;=\; \bigvee_{i\in I} (\mu_i(x) * \nu_i(y)) \tag{24}$$

Considering (20) as the system of fuzzy relational equations

$$\varrho[\mu_i] \;=\; \nu_i \qquad\qquad (i \in I),$$

where the fuzzy sets μ_i and ν_i are given and solution of the system in the form of a fuzzy relation ϱ has to be constructed, then it is well known that ϱ_U is always a solution if there exists a solution at all [3]. In this case, ϱ_U is the greatest solution. ϱ_L might not be a solution, even if there exists a solution of the system.

In some applications of approximate reasoning, especially in fuzzy control, the if-then-rules (20) are intended to describe a functional dependence between the variable ξ and η. In this case we would expect the fuzzy relation ϱ_U or ϱ_L constructed from these rules to behave like a fuzzy function. But what do we mean by a fuzzy function? A fundamental property of a function is that it is a relation that does never assign two or more elements from the codomain to one element of the domain. Thus we need the concept of a one-element (fuzzy) set again which means that we have to assume suitable equality relations on the domains X of the variable ξ and Y of the variable η. In our considerations we will usually choose the equality relation (4) on X respectively Y that is induced by the fuzzy sets appearing in the rules. But for the moment we do not need this assumption, we just have to assume that there is an equality relation E on the set X and an equality relation F on the set Y. We require a similar extensionality property from a fuzzy relation as from fuzzy sets (compare also [4, 10]).

Definition 5 *Let E and F be equality relations on the sets X and Y, respectively. A fuzzy relation $\varrho \in L^{X\times Y}$ is called extensional w.r.t. E and F if*

(i) $\varrho(x,y) * E(x,x') \le \varrho(x',y)$ *and*

(ii) $\varrho(x,y) * F(y,y') \le \varrho(x,y')$

hold.

The extensionality of a fuzzy relation simple says (in fuzzy terms) that we can replace equal elements in the first and second place of the relation.

A partial function is a relation that assigns to each element of its domain at most one element of the codomain. In other words, if the (crisp) relation $R \subseteq X \times Y$ is a partial function then we know

$$(x,y) \in R \text{ and } (x,y') \in R \text{ implies } y = y'.$$

The following definition extends this notion to fuzzy relations.

Definition 6 *Let E and F be equality relations on the sets X and Y, respectively. An extensional fuzzy relation $\varrho \in L^{X \times Y}$ is called a partial fuzzy function if*

$$\varrho(x,y) * \varrho(x,y') \;\leq\; F(y,y') \tag{25}$$

holds for all $x \in X$ and for all $y, y' \in Y$.

For a fuzzy function we still need the definition of a fully defined fuzzy relation.

Definition 7 *Let E and F be equality relations on the sets X and Y, respectively. An extensional fuzzy relation $\varrho \in L^{X \times Y}$ is called a fully defined if*

$$\bigvee_{x \in X} \varrho(x,y) \;=\; \mathbb{1}$$

holds for all $y \in Y$.

Definition 8 *Let E and F be equality relations on the sets X and Y, respectively. An extensional fuzzy relation $\varrho \in L^{X \times Y}$ is called a fuzzy function if it is a fully defined partial fuzzy function.*

Definition 9 *Let E and F be equality relations on the sets X and Y, respectively, and let $f : X \longrightarrow Y$ be an ordinary partial function. f is called extensional if*

$$E(x,x') \;\leq\; F(f(x), f(x'))$$

holds for all $x, x' \in dom(X)$, where is $dom(f) \subseteq X$ denotes the domain of f, i.e. the set of elements $x' \in X$ for which $f(x)$ is defined.

Extensionality of a partial function means that it maps indistinguishable elements to indistinguishable elements. In general, the characteristic function of an extensional partial function f will not be extensional as a fuzzy relation (compare Definition 5). Therefore, we define the following fuzzy relation induced by f:

$$\varrho_f(x,y) \;=\; \bigvee_{x' \in \mathrm{dom}(f)} E(x,x') * F(y, f(x')).$$

Lemma 1 *Let E and F be equality relations on the sets X and Y, respectively, and let $f : X \longrightarrow Y$ be an ordinary extensional partial function. Then the fuzzy relation ϱ_f is a partial fuzzy function.*

Proof. We first prove the extensionality of ϱ_f.

$$\begin{aligned}
\varrho_f(x,y) * E(x,x') \;&=\; \bigvee_{x'' \in \mathrm{dom}(f)} (E(x,x'') * F(y, f(x'')) * E(x,x')) \\
&\leq\; \bigvee_{x'' \in \mathrm{dom}(f)} (E(x',x'') * F(y, f(x''))) \\
&=\; \varrho_f(x',y)
\end{aligned}$$

$$
\begin{aligned}
\varrho_f(x,y) * F(y,y') &= \bigvee_{x' \in \mathrm{dom}(f)} (E(x,x') * F(y,f(x'')) * F(y,y')) \\
&\leq \bigvee_{x' \in \mathrm{dom}(f)} (E(x,x') * F(y', f(x'))) \\
&= \varrho_f(x,y')
\end{aligned}
$$

Thus ϱ_f is an extensional fuzzy relation. Furthermore, we have

$$
\begin{aligned}
&\varrho_f(x,y) * \varrho_f(x,y') \\
&= \bigvee_{x',x'' \in \mathrm{dom}(f)} (E(x,x') * F(y,f(x'')) * E(x,x'') * F(y',f(x''))) \\
&\leq \bigvee_{x',x'' \in \mathrm{dom}(f)} (E(x',x'') * F(y,f(x')) * F(y',f(x''))) \\
&\leq \bigvee_{x',x'' \in \mathrm{dom}(f)} (F(f(x'),f(x'')) * F(y,f(x')) * F(y',f(x''))) \\
&\leq F(y,y')
\end{aligned}
$$

so that ϱ_f is a partial fuzzy function. □

When we start with an extensional function instead of an extensional partial function, the term for ϱ_f can be simplified and ϱ_f is a fuzzy function.

Lemma 2 *Let E and F be equality relations on the sets X and Y, respectively, and let $f : X \longrightarrow Y$ be an ordinary extensional function. Then*

$$\varrho_f(x,y) = F(y, f(x))$$

holds.

Proof.

$$
\begin{aligned}
\varrho_f(x,y) &= \bigvee_{x' \in X} (E(x,x') * F(y,f(x'))) \\
&\leq \bigvee_{x' \in X} (F(f(x),f(x')) * F(y,f(x'))) \\
&\leq F(y,f(x))
\end{aligned}
$$

$$
\begin{aligned}
\varrho_f(x,y) &= \bigvee_{x' \in X} (E(x,x') * F(y,f(x'))) \\
&\geq E(x,x) * F(y,f(x)) \\
&= F(y,f(x))
\end{aligned}
$$

□

Corollary 3 *Let E and F be equality relations on the sets X and Y, respectively, and let $f : X \longrightarrow Y$ be an ordinary extensional function. Then the fuzzy relation ϱ_f is a fuzzy function.*

Proof. By Lemma 1 we have that ϱ_f is a partial fuzzy function. Using Lemma 2 we have

$$\bigvee_{y\in Y} \varrho_f(x,y) \;=\; \bigvee_{y\in Y} F(f(x),y) \;=\; \mathbb{1}$$

so that ϱ_f is fully defined. □

Let us now return to the fuzzy relations ϱ_U and ϱ_L that are induced by the collection of if-then-rules (20). The following theorem shows that they are extensional when the considered fuzzy sets are extensional.

Theorem 4 *Let E and F be equality relations on X and Y, respectively. If the fuzzy sets μ_i and ν_i $(i \in I)$ in* (20) *are extensional w.r.t. E and F, respectively, then the fuzzy relations ϱ_L and ϱ_U defined in* (23) *and* (24) *are also extensional.*

Proof. The extensionality of ϱ_L follows directly from its the definition and the extensionality of the fuzzy sets μ_i and ν_i.

Using the GL-monoid law $\alpha * (\alpha \to \beta) = \alpha \wedge \beta$ (see for instance [5]), we obtain

$$\begin{aligned} \nu_i(y) &\geq \mu_i(x) \wedge \nu_i(y) \\ &= \mu_i(x) * (\mu_i(x) \to \nu_i(y)) \\ &\geq \mu_i(x') * E(x,x') * (\mu_i(x) \to \nu_i(y)). \end{aligned}$$

By residuation and then taking the infimum over $i \in I$ we get

$$\bigwedge_{i\in I} (E(x,x') * (\mu_i(x) \to \nu_i(y))) \;\leq\; \bigwedge_{i\in I} (\mu_i(x') \to \nu_i(y)))$$

which proves $E(x,x') * \varrho_U(x,y) \leq \varrho_U(x',y)$. Using the GL-monoid law $(\alpha \to \beta) * \gamma \leq \alpha \to (\beta * \gamma)$ (see for instance [12]) we obtain

$$\begin{aligned} \varrho_U(x,y) * F(y,y') &\leq \bigwedge_{i\in I} ((\mu_i(x) \to \nu_i(y)) * F(y,y')) \\ &\leq \bigwedge_{i\in I} (\mu_i(x) \to (\nu_i(y) * F(y,y'))) \\ &\leq \bigwedge_{i\in I} (\mu_i(x) \to \nu_i(y')) \end{aligned}$$

so that ϱ_U is also extensional. □

Now we take a look at the fuzzy relation ϱ_L and examine when it provides a solution to the system of fuzzy relational equations induced by the if-then-rules.

Theorem 5 *Let the fuzzy sets μ_i and ν_i appearing in the if-then-rules* (20) *be normal. (A fuzzy set μ is normal if there exists an x such that $\mu(x) = \mathbb{1}$ holds.) The fuzzy relation ϱ_L defined in Equation* (24) *is a solution to the system of fuzzy relational equations $\varrho[\mu_i] = \nu_i$ $(i \in I)$ if and only if*

$$(\forall i, j \in I)\left(\bigvee_{x \in X} (\mu_i(x) * \mu_j(x)) \;\leq\; \bigwedge_{y \in Y} (\nu_i(y) \leftrightarrow \nu_j(y))\right) \tag{26}$$

holds.

Proof. From the definition of ϱ_L it is obvious that $\varrho_L[\mu_i] \geq \nu_i$ is always satisfied when the fuzzy sets are normal. So the only interesting question is whether $\varrho_L[\mu_i] \leq \nu_i$ holds for all $i \in I$.

$$(\forall i \in I)\,(\varrho_L[\mu_i] \leq \nu_i)$$

$$\iff \quad (\forall i \in I)(\forall y \in Y)\left(\bigvee_{j \in I, x \in X} (\mu_i(x) * \mu_j(x) * \nu_j(y)) \leq \nu_i(y)\right)$$

$$\iff \quad (\forall i, j \in I)(\forall x \in X)(\forall y \in Y)\,(\mu_i(x) * \mu_j(x) * \nu_j(y) \leq \nu_i(y)) \tag{27}$$

$$\iff \quad (\forall i, j \in I)(\forall x \in X)(\forall y \in Y)\,(\mu_i(x) * \mu_j(x) \leq \nu_j(y) \rightarrow \nu_i(y))$$

$$\overset{*\ \text{commut.}}{\iff} \quad (\forall i, j \in I)(\forall x \in X)(\forall y \in Y)\,(\mu_i(x) * \mu_j(x) \leq \nu_j(y) \leftrightarrow \nu_i(y))$$

$$\iff \quad (\forall i, j \in I)\left(\bigvee_{x \in X} (\mu_i(x) * \mu_j(x)) \leq \bigwedge_{y \in Y} (\nu_i(y) \leftrightarrow \nu_j(y))\right)$$

The following theorem justifies why Mamdani fuzzy controllers are usually associated with the notion of fuzzy functions. The fuzzy relation ϱ_L induced by the Mamdani fuzzy controller with the rule base (20) is a solution of the corresponding system of fuzzy relational equations if and only if ϱ_L is a partial fuzzy function.

Theorem 6 *Let the fuzzy sets ν_i appearing in the if-then-rules* (20) *be normal. The fuzzy relation ϱ_L defined in Equation* (25) *is a solution to the system of fuzzy relational equations $\varrho[\mu_i] = \nu_i$ $(i \in I)$ if ϱ_L has the property* (25) *of a partial fuzzy function w.r.t. to the equality F on Y induced by the fuzzy sets ν_i, $(i \in I)$ using Equation* (4).

Proof. If ϱ_L has the property (24), then $\varrho_L(\tilde{x}, \tilde{y}) * \varrho_L(\tilde{x}, \tilde{y}') \leq F(\tilde{y}, \tilde{y}')$ holds for all $\tilde{x} \in X$ and all $\tilde{y}, \tilde{y}' \in Y$, i.e.

$$(\forall \tilde{x} \in X)(\forall \tilde{y}, \tilde{y}' \in Y)$$

$$\left(\bigvee_{\tilde{i},\tilde{j}\in I}\Big(\mu_{\tilde{i}}(\tilde{x}) * \mu_{\tilde{j}}(\tilde{x}) * \nu_{\tilde{i}}(\tilde{y}) * \nu_{\tilde{j}}(\tilde{y}')\Big) \leq \bigwedge_{\tilde{k}\in I}(\nu_{\tilde{k}}(\tilde{y}) \leftrightarrow \nu_{\tilde{k}}(\tilde{y}'))\right)$$

$\iff$

$$(\forall \tilde{i},\tilde{j},\tilde{k} \in I)(\forall \tilde{x} \in X)(\forall \tilde{y},\tilde{y}' \in Y)$$
$$\Big(\mu_{\tilde{i}}(\tilde{x}) * \mu_{\tilde{j}}(\tilde{x}) * \nu_{\tilde{i}}(\tilde{y}) * \nu_{\tilde{j}}(\tilde{y}') \leq \nu_{\tilde{k}}(\tilde{y}) \leftrightarrow \nu_{\tilde{k}}(\tilde{y}')\Big)$$

$\iff$

$$(\forall \tilde{i},\tilde{j},\tilde{k} \in I)(\forall \tilde{x} \in X)(\forall \tilde{y},\tilde{y}' \in Y)$$
$$\Big(\mu_{\tilde{i}}(\tilde{x}) * \mu_{\tilde{j}}(\tilde{x}) * \nu_{\tilde{i}}(\tilde{y}) * \nu_{\tilde{j}}(\tilde{y}') \leq \nu_{\tilde{k}}(\tilde{y}) \rightarrow \nu_{\tilde{k}}(\tilde{y}')\Big)$$

$\iff$

$$(\forall \tilde{i},\tilde{j},\tilde{k} \in I)(\forall \tilde{x} \in X)(\forall \tilde{y},\tilde{y}' \in Y)$$
$$\Big(\mu_{\tilde{i}}(\tilde{x}) * \mu_{\tilde{j}}(\tilde{x}) * \nu_{\tilde{i}}(\tilde{y}) * \nu_{\tilde{j}}(\tilde{y}') * \nu_{\tilde{k}}(\tilde{y}) \leq \nu_{\tilde{k}}(\tilde{y}')\Big) \tag{28}$$

When we choose $\tilde{i} = i, \tilde{j} = j, \tilde{k} = i, \tilde{x} = x, \tilde{y}' = y$ and $\tilde{y}$ such that $\nu_{\tilde{k}}(\tilde{y}) = \nu_{\tilde{i}}(\tilde{y}) = \mathbb{1}$ in (28), we get (27) which is equivalent to the fact that ϱ_L is a solution of the system of fuzzy relational equations according to the proof of Theorem 5. $\square$

When we choose the equality relation F on Y as in Theorem 6 and the crisp equality as the equality relation on X, then the fuzzy relation ϱ_L is always extensional w.r.t. E and F according to Theorem 4. Therefore, ϱ_L is a solution to the system of fuzzy relational equations induced by the if-then-rules if ϱ_L is a partial fuzzy function.

Equation (26) in Theorem 5 can be interpreted as the requirement that the degree of non-disjointness of fuzzy sets μ_i and μ_j must not exceed the degree of equality of the corresponding fuzzy sets ν_i and ν_j. When we require the fuzzy sets μ_i $(i \in I)$ to be pairwise disjoint w.r.t. $*$, then this condition is automatically satisfied. In this case we can even provide an equality relation on X such that the fuzzy sets correspond to the extensional hulls of single points.

With this stronger assumption we can establish another connection between ϱ_L, ϱ_U and fuzzy functions.

Theorem 7 *Let E and F be equality relations on X and Y, respectively, such that the fuzzy sets μ_i and ν_i correspond to the extensional hulls of the points x_i and y_i, respectively ($i \in I$). If the ordinary partial function $f(x_i) = y_i$ is extensional w.r.t. E and F, then the fuzzy relation ϱ_L is a partial fuzzy function and $\varrho_L = \varrho_f$ holds.*

Proof. By definition of ϱ_f and ϱ_L we have

$$\begin{aligned} \varrho_L(x,y) &= \bigvee_{i \in I} (\mu_i(x) * \nu_i(y)) \\ &= \bigvee_{i \in I} (E(x,x_i) * F(y,y_i)) \\ &= \varrho_f(x,y) \end{aligned}$$

Since ϱ_f is a partial fuzzy function according to Lemma 1, ϱ_L is also a partial fuzzy function. □

In the same context the fuzzy relation ϱ_U is a fully defined fuzzy relation, but usually not a partial fuzzy function. The proof is obvious from the definition of ϱ_U.

Corollary 4 *Let E and F be equality relations on X and Y, respectively, such that the fuzzy sets μ_i and ν_i correspond to the extensional hulls of the points x_i and y_i, respectively ($i \in I$). Then*

$$\bigvee_{y \in Y} \varrho_U(x,y) = \mathbb{1}$$

holds.

Let us now formulate an important theorem and discuss its consequences for fuzzy control after the proof.

Theorem 8 *Let E and F be equality relations on X and Y, respectively, and let $f : X \longrightarrow Y$ be an (ordinary) extensional function. Let $\{x_i \mid i \in I\} \subseteq X$ be a set of elements of X and let f_I denote the (ordinary) partial function defined by $f_I(x_i) = f(x_i)$ for $i \in I$. Let μ_i denote the extensional hull of the point x_i w.r.t. E, and let ν_i denote the extensional hull of the point $f(x_i)$ w.r.t. F. Then*

$$\varrho_L = \varrho_{f_I} \le \varrho_f \le \varrho_U$$

holds.

Proof. The first equation was proved in Theorem 7. The first inequality holds by definition. We only have to prove the last inequality. From

$$E(x,x_i) * F(y,f(x)) \le F(f(x),f(x_i)) * F(y,f(x)) \le F(y,f(x_i))$$

we obtain by residuation

$$F(y, f(x)) \leq E(x, x_i) \to F(y, f(x_i)) = \mu_i(x) \to \nu_i(y)$$

for all $i \in I$. With Lemma 2 we finally have

$$\varrho_f(x, y) = F(y, f(x)) \leq \bigwedge_{i \in I} (\mu_i(x) \to \nu_i(y)) = \varrho_U(x, y).$$

□

We can interpret Theorem 8 in the context of fuzzy control in the following way. Fuzzy control aims at determining an (unknown) control function $f : X \to Y$. This function is described by if-then-rules of the form (20). The fuzzy sets μ_i and ν_i appearing in the rules are considered as extensional hulls of single points x_i and y_i. Thus the rules specify the partial control function f_I. Of course, the underlying equality relations must be related to the control function in the sense that the control function f is extensional. This means simply that we choose narrower fuzzy sets where exact values are quite important for a good control and wider fuzzy sets where even a rough controller output provides a reasonable control. Since the partial control function f_I does not specify controller outputs for all inputs, we have to take the information encoded in the fuzzy sets (or in the equality relations) into account. Therefore we consider the extensional hull of the partial control function f_I that is equal to the fuzzy relation ϱ_L, in other words, to the Mamdani-type fuzzy control scheme. This provides a lower approximation for the extensional hull of the (unknown) control function f. The fuzzy relation ϱ_U can be seen as an upper approximation of the extensional hull of f. Similar considerations on lower and upper approximations of functions or relations can be found in [14].

Let us conclude this section with a corollary following from Theorem 7 and Corollary 4.

Corollary 5 *Let E and F be equality relations on X and Y, respectively, such that the fuzzy sets μ_i and ν_i correspond to the extensional hulls of the points x_i and y_i, respectively ($i \in I$). Let the ordinary partial function $f(x_i) = y_i$ be extensional w.r.t. E and F. If*

$$\varrho_L = \varrho_U$$

holds, then ϱ_L and ϱ_U are fuzzy functions.

5 Conclusions

In this paper we have provided a possible to the concept of a fuzzy function and established a strong connection to fuzzy control. The Mamdani fuzzy controller can be interpreted in the sense of a partial fuzzy function. This interpretation is only admissible when the fuzzy sets satisfy certain restriction, i.e. when they can be seen as fuzzy points. This view clarifies a rational behind fuzzy control and why not arbitrary fuzzy sets are used in fuzzy control.

As a final remark we point that the equality relations are not only important for the interpretation of the fuzzy sets as fuzzy points, but also characterize an indistinguishability that is inherent in any fuzzy system. In [12] it was shown in quite general term that the output of a fuzzy system does not change when the input is replaced by its extensional hull. And the output is always an extensional fuzzy set. In this sense, the indistinguishability inherent in fuzzy partitions cannot be overcome.

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