Similarity in Fuzzy Reasoning

Frank Klawonn
Department of Computer Science
University of Braunschweig
38106 Braunschweig, Germany

Juan Luis Castro
Dpto. Ciencias de la Computación e Intelligencia Artificial
ETSI Ingeniería Informática
Universidad de Granada, 18071
Granada, Spain

Abstract

Fuzzy set theory is based on a 'fuzzification' of the predicate $\in$ (element of), the concept of membership degrees is considered as fundamental. In this paper we elucidate the connection between indistinguishability modelled by fuzzy equivalence relations and fuzzy sets. We show that the indistinguishability inherent to fuzzy sets can be computed and that this indistinguishability cannot be overcome in approximate reasoning.

For our investigations we generalize from the unit interval as the basis for fuzzy sets, to the framework of GL–monoids that can be understood as a generalization of MV–algebras. Residuation is a basic concept in GL–monoids and many proofs can be formulated in a simple and clear way instead of using special properties of the unit interval.

1 Introduction

Fuzzy set theory is based on the idea that many non–mathematical properties cannot be described in terms of crisp sets comprising those elements that fulfill a given property. Therefore the notion of membership is considered as a gradual property for fuzzy sets.

Similarity or indistinguishability is another important concept for which a crisp model is often inadequate. A possible formalization of similarity or indistinguishability is the notion of fuzzy equivalence or equality relation.

In this paper we examine the relations between fuzzy equivalence relations and fuzzy sets. As the underlying formal framework, we abstract from the unit interval to GL–monoids – a structure in which residuation plays a fundamental role. Examples for GL–monoids are complete MV–algebras or the unit interval endowed with the usual ordering and a left continuous t–norm.

Section 2 provides the formal basis for this paper by a brief review of the definition of GL–monoids including some useful properties and examples as well as the definitions of fuzzy equivalence relations and extensionality which requires that a fuzzy set behaves well with respect to a given fuzzy equivalence relation.

In Section 3 we take a closer look at the extensionality property and show how a suitable fuzzy equivalence relation, which describes the implicit indistinguishability in fuzzy sets, can be derived from a given collection of fuzzy sets.

Our main results are presented in Section 4, where we prove that the indistinguishability inherent to fuzzy sets cannot be overcome by the usual approximate reasoning schemes so that fuzzy equivalence relations provide a useful description of this indistinguishability.

2 GL–Monoids, Fuzzy Sets and Similarity Relations

In this Section we briefly introduce the notions on which our considerations are based. The underlying structure for our investigations are GL–monoids. From [8], we recall the definition of a GL–monoid and some properties which we will use.

Definition 2.1 $(L, \leq, *)$ *is a GL–monoid iff*

1. $(L, \leq)$ *is a complete lattice,*

2. *$(L, *)$ is a commutative monoid, i.e. the operation $* : L \times L \longrightarrow L$ is associative and commutative and has a unit $\mathbb{1} \in L$,*

3. *$(L, *)$ has a zero element $\mathbf{0} \in L$ fulfilling $\alpha * \mathbf{0} = \mathbf{0}$,*

4. *$*$ is isotonic, i.e.*
$$\alpha \leq \beta \implies \alpha * \gamma \leq \beta * \gamma,$$

5. *$(L, \leq, *)$ is integral, i.e. $\mathbb{1} = \bigvee L$ is also the universal upper bound of L,*

6. *$(L, \leq, *)$ is the dual of a divisibility monoid, i.e.*
$$\alpha \leq \beta \text{ implies the existence of } \gamma \in L \text{ such that } \alpha = \beta * \gamma,$$

7. *$(L, \leq, *)$ is residuated, meaning that there exists a binary operation $\rightarrow$ on L satisfying*
$$\alpha * \beta \leq \gamma \iff \alpha \leq \beta \rightarrow \gamma, \tag{1}$$

8. *the infinite distributive law holds, i.e.*
$$\alpha * \bigvee_{i \in I} \beta_i \;=\; \bigvee_{i \in I} (\alpha * \beta_i).$$

For the rest of this paper we assume $(L, \leq, *)$ to be a GL–Monoid. L will be considered as the set of "truth"–values of a many–valued logic.

The binary operation $\rightarrow$ is uniquely determined by the adjunction property (1):
$$\alpha \rightarrow \beta \;=\; \bigvee \{\lambda \in L \mid \alpha * \lambda \leq \beta\}. \tag{2}$$
$\rightarrow$ can be viewed as the valuation function for the implication (associated with the conjunction $*$). From this implication we can derive in a canonical way a valuation for the negation by defining $\neg\alpha = \alpha \rightarrow \mathbf{0}$. Note that in a GL–monoid the zero element of $*$ is also the universal lower bound, i.e. $\mathbf{0} = \bigwedge L$. We define the biimplication $\leftrightarrow : L \times L \longrightarrow L$ by
$$\alpha \leftrightarrow \beta \;=\; (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha).$$

Interesting for applications is the case when L is the unit interval with the usual linear ordering. $*$ can be any continuous t–norm (a commutative, associative, non–decreasing binary operation on $[0,1]$ having 1 as unit). $*$ is understood as an alternative to the lattice operation $\wedge$ (in the case of the unit interval simply min) for the valuation function of a conjunction.

Note that a GL–monoid is a complete MV–algebra (see, e.g., [1, 4]) if and only if the 'negation' $\alpha \to \mathbf{0}$ is involutory, i.e. $\neg\neg\alpha = (\alpha \to \mathbf{0}) \to \mathbf{0} = \alpha$. However, for practical applications with the unit interval as the underlying lattice L, considering only MV–algebras is very restrictive, since $\alpha * \beta = \max\{\alpha + \beta - 1, 0\}$ is the only choice for the operation $*$ up to isomorphism. This holds for the following reason. Because $\neg$ is an involution and non–increasing, in accordance to the isotonicity of $*$, $\neg$ is also continuous. For MV–algebras, $*$ satisfies also the equation (cf. Lemma 1.4(6) of [8])

$$\alpha * \bigwedge_{i \in I} \beta_i \;=\; \bigwedge_{i \in I} (\alpha * \beta_i),$$

so that together with the infinite distributivity of $*$ over arbitrary joins, we obtain that $*$ is also continuous. This implies that $\to$ is continuous, since we have (cf. Lemma 1.4(3) of [8])

$$\alpha \to \beta \;=\; \neg(\alpha * \neg\beta) \;=\; (\alpha * (\beta \to \mathbf{0})) \to \mathbf{0}.$$

But if the unit interval is the underlying lattice of the GL–monoid whose operation $\to$ is continuous, then $([0,1], \leq, *)$ is isomorphic to the GL–monoid (MV–algebra) $([0,1], \leq, *_L)$ with

$$\alpha *_L \beta \;=\; \max\{\alpha + \beta - 1, 0\} \qquad\qquad \text{(Łukasiewicz conjunction)}$$

(see, e.g., [19]).

Example 2.2. Let $L = [0,1]$ be the unit interval with the usual ordering. Then

$$\alpha \leftrightarrow \beta \;=\; \max\{\alpha, \beta\} \to \min\{\alpha, \beta\}$$

holds [22].

The following table provides some examples for the operation $*$ and the induced operations $\to, \leftrightarrow$, and $\neg$ (cf. [12, 18]).

$\alpha * \beta$	$\max\{\alpha+\beta-1, 0\}$	$\min\{\alpha, \beta\}$	$\alpha \cdot \beta$
$\alpha \to \beta$	$\min\{1-\alpha+\beta, 1\}$	$\begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha \leq \beta \\ \frac{\beta}{\alpha} & \text{otherwise} \end{cases}$
$\alpha \leftrightarrow \beta$	$1 - \lvert\alpha - \beta\rvert$	$\begin{cases} 1 & \text{if } \alpha = \beta \\ \min\{\alpha, \beta\} & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha = \beta \\ \frac{\min\{\alpha,\beta\}}{\max\{\alpha,\beta\}} & \text{otherwise} \end{cases}$
$\neg\alpha$	$1 - \alpha$	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$

If we choose $* = \wedge$, then $(L, \leq, *)$ is a Heyting algebra, where we obtain for $L = [0, 1]$ the Gödel implication (see the third column in the table above) as the operation $\to$.

Let us collect some useful properties of GL–monoids which we need in this paper.

Lemma 2.3. *In any GL–monoid* $(L, \leq, *)$, *the following properties are valid:*

(a) $\alpha \wedge \beta = \alpha * (\alpha \to \beta)$,

(b) $\alpha * \bigwedge_{i \in I} \beta_i \leq \bigwedge_{i \in I} (\alpha * \beta_i)$,

(c) $(\alpha \to \beta) * (\beta \to \gamma) \leq \alpha \to \gamma$,

(d) $(\alpha \leftrightarrow \beta) * (\beta \leftrightarrow \gamma) \leq \alpha \leftrightarrow \gamma$,

(e) $\alpha \leq \beta \;\Rightarrow\; \beta \to \gamma \leq \alpha \to \gamma$,

(f) $(\alpha * \beta) \to \gamma = \alpha \to (\beta \to \gamma)$,

(g) $\alpha \leq \beta \;\Rightarrow\; \gamma \to \alpha \leq \gamma \to \beta$,

(h) $(\alpha \to \beta) * \gamma \leq \alpha \to (\beta * \gamma)$,

(i) $\alpha \to (\beta \wedge \gamma) = (\alpha \to \beta) \wedge (\alpha \to \gamma)$,

(j) $(\alpha_1 \leftrightarrow \beta_1) \wedge (\alpha_2 \leftrightarrow \beta_2) \leq (\alpha_1 \wedge \alpha_2) \leftrightarrow (\beta_1 \wedge \beta_2)$.

Proof. (a), (f), and (i) are proved in the Lemmas 1.1(2) and 1.2(3) of [8].

(b) The monotonicity of $*$ implies $\alpha * \bigwedge_{i \in I} \beta_i \leq \alpha * \beta_j$ for all $j \in I$.

(c) In accordance to the adjunction property, we have to prove that

$$\alpha * (\alpha \to \beta) * (\beta \to \gamma) \;\leq\; \gamma.$$

Using (a), we obtain:

$$\begin{aligned}\alpha * (\alpha \to \beta) * (\beta \to \gamma) &= (\alpha \wedge \beta) * (\beta \to \gamma)\\ &\leq \beta * (\beta \to \gamma)\\ &= \beta \wedge \gamma\\ &\leq \gamma.\end{aligned}$$

(d) By applying (b) and (c), we get:

$$\begin{aligned}(\alpha \leftrightarrow \beta) * (\beta \leftrightarrow \gamma) &= ((\alpha \to \beta) \wedge (\beta \to \alpha)) * ((\beta \to \gamma) \wedge (\gamma \to \beta))\\ &\leq ((\alpha \to \beta) * (\beta \to \gamma)) \wedge ((\gamma \to \beta) * (\beta \to \alpha))\\ &\leq (\alpha \to \gamma) \wedge (\gamma \to \alpha)\\ &= \alpha \leftrightarrow \gamma.\end{aligned}$$

(e) and (g) are direct consequences of the isotonicity of $*$ and equation (2).

(h) In accordance to the adjunction property, we have to prove that

$$\alpha * (\alpha \to \beta) * \gamma \;\leq\; \beta * \gamma.$$

Due to (a), the left hand side of this inequality is simply $(\alpha \wedge \beta) * \gamma$ which is less than or equal to $\beta * \gamma$ because of the isotonicity of $*$.

(j) Using (i) and (e), we obtain:

$$\begin{aligned}(\alpha_1 \wedge \alpha_2) \leftrightarrow (\beta_1 \wedge \beta_2) &= ((\alpha_1 \wedge \alpha_2) \to (\beta_1 \wedge \beta_2))\\ &\quad \wedge((\beta_1 \wedge \beta_2) \to (\alpha_1 \wedge \alpha_2))\\ &= ((\alpha_1 \wedge \alpha_2) \to \beta_1) \wedge ((\alpha_1 \wedge \alpha_2) \to \beta_2)\\ &\quad \wedge((\beta_1 \wedge \beta_2) \to \alpha_1) \wedge ((\beta_1 \wedge \beta_2) \to \alpha_2)\\ &\geq (\alpha_1 \to \beta_1) \wedge (\beta_1 \to \alpha_1)\\ &\quad \wedge(\alpha_2 \to \beta_2) \wedge (\beta_2 \to \alpha_2)\\ &= (\alpha_1 \leftrightarrow \beta_1) \wedge (\alpha_2 \leftrightarrow \beta_2).\end{aligned}$$

□

As we have already mentioned, we interpret L as the set of truth values. Thus a generalization of definitions of classical concepts is necessary. Here we restrict ourselves to the notions of subsets and equivalence relations.

Definition 2.4. *An L–fuzzy (sub)set (or simply a fuzzy set) of the set X is a mapping $\mu : X \longrightarrow L$.*

The value $\mu(x) \in L$ is understood as the degree or truth value of x being an element of the (sub)set μ.

Definition 2.5. *An L–fuzzy equivalence relation (or simply a fuzzy equivalence relation) (with respect to the operation $*$) on the set X is a mapping $E : X \times X \longrightarrow L$ satisfying the axioms:*

(E1)	$E(x,x) = 1,$	(reflexivity)
(E2)	$E(x,y) = E(y,x),$	(symmetry)
(E3)	$E(x,y) * E(y,z) \leq E(x,z).$	(transitivity)

Note that for $L = \{0,1\}$, E is a fuzzy equivalence relation if and only if it is the characteristic function of an ordinary equivalence relation. Depending on the choice of the operation $*$, sometimes E is also called a similarity relation [24], indistinguishability operator [21], fuzzy equality (relation) [9, 16] or proximity relation [6].

Example 2.6.

(i) If $\approx$ is an equivalence relation on X, then its characteristic function

$$E_R(x,y) = \begin{cases} 1 & \text{if } x \approx y, \\ 0 & \text{otherwise,} \end{cases}$$

is a fuzzy equivalence relation with respect to any choice of the operation $*$. Especially the characteristic function of the crisp equality on X is a fuzzy equivalence relation.

(ii) Let $L = [0,1]$ and let $*$ be the Łukasiewicz conjunction. Then E is a fuzzy equivalence relation on X with respect to $*$ if and only if $1 - E$ is a pseudo–metric on X. Thus pseudo–metrics bounded by one and fuzzy equivalence relations with respect to the Łukasiewicz conjunction are dual concepts. If a pseudo–metric δ is not bounded by one, we can enforce this property by considering the pseudo–metric $\bar{\delta} = \min\{\delta(x,y), 1\}$, which coincides with δ for "small" distances. Thus any pseudo–metric induces a fuzzy equivalence relation on X with respect to the Łukasiewicz conjunction by

$$E(x,y) \;=\; 1 - \bar{\delta}(x,y) \;=\; 1 - \min\{\delta(x,y), 1\}.$$

(iii) We obtain the same duality as in (ii) when we replace the notion of metric by ultra–metrics and the Łukasiewicz conjunction by the minimum.

When we interpret an ordinary equivalence relation $\approx$ on the set X in the sense that equivalent elements cannot be distinguished or may be identified, then only those subsets $M \subseteq X$ satisfying

$$x \in M \text{ and } x \approx y \;\Rightarrow\; y \in M \tag{3}$$

"behave well" with respect to $\approx$. In other words, a subset M fulfills condition (3) if and only if it equals a union of equivalence classes of $\approx$. The following definition generalizes the axiom (3) for fuzzy equivalence relations and fuzzy sets.

Definition 2.7. *A fuzzy set $\mu \in L^X$ is called extensional w.r.t. the fuzzy equivalence relation E on X iff*

$$\mu(x) * E(x,y) \;\leq\; \mu(y)$$

holds for all $x, y \in X$.

If a fuzzy set μ is not extensional with respect to the considered fuzzy equivalence relation E, we may consider instead of μ the smallest extensional fuzzy set which contains μ.

Definition 2.8. *Let E be a fuzzy equivalence relation on X and let $\mu \in L^X$. The fuzzy set*

$$\hat{\mu} \;=\; \bigwedge\{\nu \mid \mu \leq \nu \text{ and } \nu \text{ is extensional w.r.t. } E\}$$

is called the extensional hull of μ w.r.t. E.

Proposition 2.9. *Let E be a fuzzy equivalence relation on X and let $\mu \in L^X$. Then*

(i) $\hat{\mu}(x) \ = \ \bigvee\{\mu(y) * E(x,y) \mid y \in X\}$*,*

(ii) $\hat{\mu}$ *is extensional with respect to* μ*,*

(iii) $\hat{\hat{\mu}} = \hat{\mu}$*.*

Proof.

(i) Let us abbreviate the right hand side of (i) by $\tilde{\mu}$. $\tilde{\mu}$ is extensional w.r.t. E because

$$\begin{aligned} \tilde{\mu}(x) * E(x,y) &= E(x,y) * \bigvee\{\mu(z) * E(y,z) \mid z \in X\} \\ &= \bigvee\{\mu(z) * E(x,y) * E(y,z) \mid z \in X\} \\ &\leq \bigvee\{\mu(z) * E(y,z) \mid z \in X\} \\ &= \tilde{\mu}(y). \end{aligned}$$

We also have that

$$\tilde{\mu}(x) \ = \ \bigvee\{\mu(y) * E(x,y) \mid y \in X\} \ \geq \ \mu(x) * E(x,x) \ = \ \mu(x),$$

which implies $\tilde{\mu} \geq \hat{\mu}$.
In order to prove $\tilde{\mu} \leq \hat{\mu}$, let ν be an extensional fuzzy set of X w.r.t. E such that $\nu \geq \mu$. Since

$$\nu(x) \ \geq \ \nu(y) * E(x,y) \ \geq \ \mu(y) * E(x,y)$$

holds for any $y \in X$, we have also $\tilde{\mu} \leq \hat{\mu}$.

(ii) In the proof of (i) we have already shown that $\tilde{\mu} = \hat{\mu}$ is extensional w.r.t. E.

(iii) follows directly from (ii) and the definition of $\hat{\mu}$. □

Proposition 2.9(i) states that $\hat{\mu}$ can be interpreted as the union of all elements that are equivalent with respect to E to at least one of the elements of μ. In [10, 12] extensional fuzzy sets are called $*$–eigenvectors of E. In

[12] it is proved, for the case, $L = [0,1]$ that a fuzzy set μ on the set X is extensional (a $*$–eigenvector of the fuzzy equivalence relation E) if and only if it is a "generator" of E, meaning that

$$\mu(x) \leftrightarrow \mu(y) \ \geq \ E(x,y) \tag{4}$$

holds for all $x, y \in X$. In the more general framework of GL–monoids we can provide a simpler proof of this fact using only the adjunction property.

Theorem 2.10. *Let E be a fuzzy equivalence relation on X and let $\mu \in L^X$. Then μ is extensional w.r.t. E if and only if* (4) *holds for all $x, y \in X$.*

Proof. Let μ be extensional w.r.t. E. According to the adjunction property we can rewrite the extensionality condition $\mu(x) * E(x,y) \leq \mu(y)$ in the form $E(x,y) \leq \mu(x) \to \mu(y)$. Therefore, we have

$$\begin{aligned} E(x,y) &= E(x,y) \wedge E(y,x) \\ &\leq (\mu(x) \to \mu(y)) \wedge (\mu(y) \to \mu(x)) \\ &= \mu(x) \leftrightarrow \mu(y). \end{aligned}$$

To prove the other implication, let us assume that (4) holds. This implies $E(x,y) \leq \mu(x) \leftrightarrow \mu(y) \leq \mu(x) \to \mu(y)$. Again by the adjunction property, we obtain $\mu(x) * E(x,y) \leq \mu(y)$. □

Example 2.11. We can define the extensional hull $\widehat{M}$ of a crisp subset M of X w.r.t. any fuzzy equivalence relation E as the extensional hull w.r.t. E of its characteristic function

$$\mu_M(x) \ = \ \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise,} \end{cases}$$

i.e. $\widehat{M} = \widehat{\mu}_M$.

Let $X = \mathbb{R}$, $L = [0,1]$, and let $*$ be the Łukasiewicz conjunction. Let us consider the fuzzy equivalence relation induced by the usual metric on the real numbers, i.e. $E(x,y) = 1 - \min\{|x-y|, 1\}$ (cf. Example 2.6). Then, as already pointed out in [16], the extensional hull w.r.t. E of a crisp point x_0 – precisely: of the one–element set $\{x_0\}$ – is the triangular fuzzy set

$\widehat{\{x_0\}}(x) = 1 - \min\{|x_0 - x|, 1\}$, and the extensional hull w.r.t. E of the interval $[a, b]$ is the trapezoidal fuzzy set

$$\widehat{[a,b]}(x) = \begin{cases} 1 & \text{if } a \le x \le b \\ \max\{1 - a + x, 0\} & \text{if } x \le a \\ \max\{1 - x + b, 0\} & \text{if } b \le x. \end{cases}$$

By using a scaling function of the usual metric on the real numbers triangular and trapezoidal fuzzy sets with other slopes than 1 can be obtained as extensional hulls w.r.t. E of single elements and intervals, respectively [13].

3 Relations Between Fuzzy Sets and Similarity Relations

In the previous section we have seen how a crisp set induces a fuzzy set as its extensional hull with respect to a fuzzy equivalence relation. Thus, assuming the indistinguishability modelled by a fuzzy equivalence relation as a basic concept, fuzzy sets can be viewed as induced concepts, i.e. we obtain membership degrees starting from (fuzzy) indistinguishability. In this section we will take a closer look at the connection between fuzzy sets and the corresponding indistinguishability. As a main result, we show how the indistinguishability inherent to a given collection of fuzzy sets can be derived.

Theorem 3.1. *Let $\mathcal{F} \subseteq L^X$ be a set of fuzzy sets. Then*

$$E_{\mathcal{F}}(x, y) = \bigwedge_{\mu \in \mathcal{F}} (\mu(x) \leftrightarrow \mu(y)) \tag{5}$$

is the coarsest (greatest) fuzzy equivalence relation on X such that all fuzzy sets in $\mathcal{F}$ are extensional w.r.t. $E_{\mathcal{F}}$.

Proof. It is obvious that $E_{\mathcal{F}}$ is reflexive and symmetric. The transitivity of $E_{\mathcal{F}}$ follows from

$$\begin{aligned} E_{\mathcal{F}}(x, y) * E_{\mathcal{F}}(y, z) &= \left(\bigwedge_{\mu \in \mathcal{F}} (\mu(x) \leftrightarrow \mu(y)) \right) * \left(\bigwedge_{\nu \in \mathcal{F}} (\nu(y) \leftrightarrow \nu(z)) \right) \\ &\le \bigwedge_{\mu, \nu \in \mathcal{F}} \big((\mu(x) \leftrightarrow \mu(y)) * (\nu(y) \leftrightarrow \nu(z)) \big) \end{aligned}$$

$$
\begin{aligned}
&\leq \bigwedge_{\mu \in \mathcal{F}} \big((\mu(x) \leftrightarrow \mu(y)) * (\mu(y) \leftrightarrow \mu(z))\big) \\
&\leq \bigwedge_{\mu \in \mathcal{F}} (\mu(x) \leftrightarrow \mu(z)) \\
&= E_{\mathcal{F}}(x, z)
\end{aligned}
$$

where we made use of the facts (b) and (d) stated in Lemma 2.3.

The extensionality of the fuzzy sets in $\mathcal{F}$ follows directly from Theorem 2.10 and the definition of $E_{\mathcal{F}}$.

Finally, we have to show that $E_{\mathcal{F}}$ is the coarsest fuzzy equivalence relation making all fuzzy sets in $\mathcal{F}$ extensional. Let E be a fuzzy equivalence relation such that all fuzzy sets in $\mathcal{F}$ are extensional w.r.t. E. By Theorem 2.10, $E(x, y) \leq \mu(x) \leftrightarrow \mu(y)$ holds for all $\mu \in \mathcal{F}$ which implies $E(x, y) \leq E_{\mathcal{F}}(x, y)$. □

The fuzzy equivalence relation (5) was already defined in Valverde's representation theorem [23] which he proved for $L = [0, 1]$. This theorem states that $E_{\mathcal{F}}$ is a fuzzy equivalence relation if and only if there is a set $\mathcal{F}$ of fuzzy sets such that E can be written in the form (5).

The formula (5) can be interpreted in the following way. Two elements "cannot be distinguished by a (fuzzy) set" if they are either both elements of the same set or its complement, but not one in the set and the other one in the complement. Thus $\mu(x) \leftrightarrow \mu(y)$ represents the degree to which the elements x and y cannot be distinguished by the fuzzy set μ. Therefore, $E_{\mathcal{F}}(x, y)$ is the degree to which x and y cannot be distinguished by the set $\mathcal{F}$ of fuzzy sets.

The paper [11] is devoted to minimal sets of fuzzy sets that induce a fuzzy equivalence relation via (5). In [12] a geometric characterization of the set of fuzzy sets that are extensional with respect to a given fuzzy equivalence relation is described. We provide an algebraic characterization of the set of fuzzy sets that are extensional with respect to a given fuzzy equivalence relation.

Interpreting the elements of the lattice L as constant fuzzy sets, we have the following

Theorem 3.2. *Let E be a fuzzy equivalence relation on the set X. Let $\mathcal{A}_E \subseteq L^X$ denote the set of fuzzy sets that are extensional with respect to E. For all $\mu \in \mathcal{A}_E$, $\mathcal{B} \subseteq \mathcal{A}_E$, $\alpha \in L$ the following statements are valid:*

(a) $(\bigvee \mathcal{B}) \in \mathcal{A}_E$*,*

(b) $(\bigwedge \mathcal{B}) \in \mathcal{A}_E$*,*

(c) $(\alpha * \mu) \in \mathcal{A}_E$*,*

(d) $(\mu \to \alpha) \in \mathcal{A}_E$*,*

(e) $(\alpha \to \mu) \in \mathcal{A}_E$*.*

Proof. (a) follows directly from the infinite distributive law for GL–monoids. (b) is implied by Lemma 2.3(b). (c) is obvious.

For (d), we have to prove that

$$E(x,y) * (\mu(x) \to \alpha) \;\leq\; \mu(y) \to \alpha$$

for any $\mu \in \mathcal{A}_E$. In accordance to the adjunction property, this is equivalent to prove that

$$\mu(y) * E(x,y) * (\mu(x) \to \alpha) \;\leq\; \alpha. \tag{6}$$

The extensionality of μ implies that the left hand side of (6) is less than or equal to $\mu(x) * (\mu(x) \to \alpha)$ which is equal to $\alpha \wedge \mu(x)$ by Lemma 2.3(a) and therefore less than or equal to α.

In accordance to the adjunction property, it is sufficient to prove that

$$E(x,y) * \alpha * (\alpha \to \mu(x)) \;\leq\; \mu(y) \tag{7}$$

for (e). Since $\alpha * (\alpha \to \mu(x)) \leq \mu(x)$ by Lemma 2.3(a) and because of the extensionality of μ, (7) is also satisfied. □

Theorem 3.3. *Let* $\mathcal{A} \subseteq L^X$ *be such that* $\mathcal{A}$ *has properties (a),...,(e) stated in Theorem* 3.2*. Then there exists a fuzzy equivalence relation* E *on* X *for which the corresponding set of extensional fuzzy sets is exactly* $\mathcal{A}$*, i.e.* $\mathcal{A}_E = \mathcal{A}$*. Furthermore,* E *is uniquely determined by Equation* (5) *(with* $\mathcal{F} = \mathcal{A}$*).*

Proof. We first prove that $\mathcal{A} = \mathcal{A}_E$ for the fuzzy equivalence relation

$$E(x,y) \;=\; \bigwedge_{\mu \in \mathcal{A}} (\mu(x) \leftrightarrow \mu(y)).$$

By the definition of E and in accordance to Theorem 2.10, all fuzzy sets in $\mathcal{A}$ are extensional w.r.t. E, i.e. $\mathcal{A} \subseteq \mathcal{A}_E$.

In order to show the other inclusion, let $\mu \in \mathcal{A}_E$. For $z \in X$, define the fuzzy set μ_z by setting for any $x \in X$:

$$\mu_z(x) \;=\; \mu(z) * E(x,z) \;=\; \bigwedge_{\nu \in \mathcal{A}} \Big(\alpha * \big((\nu(x) \to \beta_z^{(\nu)}) \wedge (\beta_z^{(\nu)} \to \nu(x))\big)\Big),$$

where $\alpha = \mu(z)$ and $\beta_z^{(\nu)} = \nu(z)$. Due to the closure properties (b), (c), (d), (e) of $\mathcal{A}$, we have that $\mu_z \in \mathcal{A}$.

Since μ is extensional with respect to E we obtain

$$\mu_z(x) \;=\; \mu(z) * E(x,z) \;\leq\; \mu(x).$$

This implies, together with $\mu_z(z) = \mu(z) * E(z,z) = \mu(z)$, that

$$\mu \;=\; \bigvee_{z \in X} \mu_z$$

which is again an element of $\mathcal{A}$ by property (a). Thus we have also proved the other inclusion concluding $\mathcal{A} = \mathcal{A}_E$.

Let us now turn to the uniqueness of E. Let $\tilde{E}$ be a fuzzy equivalence relation such that $\mathcal{A} = \mathcal{A}_{\tilde{E}}$. In accordance to Theorem 3.1, we have $\tilde{E} \leq E$. To show that $E \leq \tilde{E}$ also holds, let $x, y \in X$. Define the fuzzy set $\nu_y(z) = \tilde{E}(z,y)$ which is obviously extensional w.r.t. $\tilde{E}$, therefore it is an element of $\mathcal{A}$ and thus also extensional w.r.t. E. Furthermore, we have $\nu_y(y) = \mathbb{1}$. We finally conclude that

$$E(x,y) \;=\; E(x,y) * \mathbb{1} \;=\; E(x,y) * \nu_y(y) \;\leq\; \nu_y(x) \;=\; \tilde{E}(x,y).$$

□

Theorems 3.2 and 3.3 establish a one–to–one correspondence between the fuzzy equivalence relations on a set X and the set of all fuzzy sets on X fulfilling the closure properties (a),...,(e) described in Theorem 3.2. (a) and (b) state that extensionality is preserved by arbitrary unions and intersections. (c) and (e) can be interpreted as some kind of cutting and lifting condition, respectively, that maintain extensionality. (d) means that extensionality is preserved under a generalized complementation, where $\alpha = \mathbf{0}$ corresponds to the usual complementation.

For MV–algebras the conditions (a),...,(e) can be simplified.

Theorem 3.4. *Let* $(L, \leq, *)$ *be a complete MV–algebra (i.e. a GL–monoid satisfying* $(\alpha \to \mathbf{0}) \to \mathbf{0} = \alpha$*). Let* $\mathcal{A} \subseteq L^X$ *be such that* $\mathcal{A}$ *has the properties:*

(i) $\mathcal{B} \subseteq \mathcal{A} \;\Rightarrow\; \bigvee \mathcal{B} \in \mathcal{A}$,

(ii) $\alpha \in L$ and $\mu \in \mathcal{A} \;\Rightarrow\; (\mu \to \alpha) \in \mathcal{A}$.

Then $\mathcal{A}$ *satisfies the closure properties (a),…,(e) stated in Theorem* 3.2.

Proof.
(a) is identical with (i). In accordance to Lemma 1.4(1) of [8], we have that

$$\left(\bigwedge_{i\in I} \alpha_i\right) \to \mathbf{0} \;=\; \bigvee_{i\in I}(\alpha_i \to \mathbf{0})$$

and therefore

$$\bigwedge \mathcal{B} \;=\; \left(\bigvee_{\nu\in\mathcal{B}} (\nu \to \mathbf{0})\right) \to \mathbf{0},$$

so that (b) also holds. Lemma 1.4(3) of [8] states that

$$\alpha \to \beta \;=\; (\alpha * (\beta \to \mathbf{0})) \to \mathbf{0}. \tag{8}$$

Thus we can rewrite $\alpha * \mu$ in the form

$$(\mu \to (\alpha \to \mathbf{0})) \to \mathbf{0} \;=\; \Big(\big(\mu * ((\alpha \to \mathbf{0}) \to \mathbf{0})\big) \to \mathbf{0}\Big) \to \mathbf{0} \;=\; \mu * \alpha,$$

so that (c) is also fulfilled. (d) is just (ii). Using Equation, (8) we can rewrite $\alpha \to \mu$ by

$$\alpha \to \mu \;=\; (\alpha * (\mu \to \mathbf{0})) \to \mathbf{0}.$$

Since we have already shown that $\mathcal{A}$ satisfies (c), we have also proved (e). □

Theorem 3.4 shows that when the underlying GL–monoid is a complete MV–algebra, the algebraic characterization of the set of extensional fuzzy sets simplifies to the closedness with respect to arbitrary unions and to the generalized complementation $\mu \to \alpha$.

In this Section we have discussed connections between fuzzy sets and fuzzy equivalence relations. For a given collection of fuzzy sets, the fuzzy equivalence relation (5) characterizes the indistinguishability inherent to these fuzzy sets. In the following section we will show that this indistinguishability cannot be overcome in typical approximate reasoning situations with fuzzy sets.

At the end of Section 2 we have explained how crisp points and sets induce fuzzy sets in the presence of a fuzzy equivalence relation. Since it is sufficient for our purposes to characterize the indistinguishability inherent to a given collection of fuzzy sets, in this paper we do not pursue the question when a given collection of fuzzy sets can be interpreted as extensional hulls of crisp elements or sets with respect to a suitable fuzzy equivalence relation. This question is treated in [13, 18, 20], for an overview see [14].

4 Similarity Relations in Fuzzy Reasoning

In [15, 18], it was shown that fuzzy control can be interpreted as interpolation in the presence of indistinguishability characterized by fuzzy equivalence relations whenever the fuzzy sets used for the fuzzy partitions satisfy some reasonable restrictions. The corresponding fuzzy equivalence relations can be computed on the basis of formula (5).

In this Section we will show that in typical applications of fuzzy reasoning – not only for fuzzy control – these fuzzy equivalence relations are of importance since they characterize an indistinguishability that cannot be overcome.

In approximate reasoning one has often to deal with if–then rules of the form

$$\text{If } \xi \text{ is } A, \text{ then } \eta \text{ is } B, \tag{9}$$

where ξ and η are variables with domains X and Y, respectively. A and B are linguistic terms like *positive big* or *approximately zero* (see, e.g., [17]). These linguistic terms are usually modelled by suitable fuzzy sets, say $\mu_A \in L^X$ and $\mu_B \in L^Y$.

The rules of the form (9) represent general knowledge about the considered problem. In the actual situation, in addition to this general knowledge, the information

$$\xi \text{ is } A' \tag{10}$$

is given, where A' is represented by the fuzzy set $\mu_{A'} \in L^X$ (or simply by $\mu \in L^X$). Of course, as a special case $\mu_{A'}$ can also stand for a crisp value

x_0 in which case $\mu_{A'}(x)$ would yield the value one for $x = x_0$ and zero otherwise. Thus, an inference scheme is needed that derives from the fuzzy sets μ_A, μ_B, appearing in the rules, and the fuzzy set $\mu_{A'}$, representing the actual information, a fuzzy set $\nu_{\text{conclusion}}$ that describes the restriction or possible values for the variable η under the given rules and the actual input information (10).

Let us first only consider a single rule of the form (9). A very convenient possibility to represent such a rule is in the form of a fuzzy relation $\varrho \in L^{X\times Y}$. Usually ρ is defined on the basis of one of the operations $\wedge, *$, and $\rightarrow$, i.e.

$$\varrho(x,y) \;=\; \varrho_{\odot}(x,y) \;=\; \mu_A(x) \odot \mu_B(y) \tag{11}$$

where $\odot \in \{\wedge, *, \rightarrow\}$. For a given input information in the form of the fuzzy set $\mu_{A'} \in L^X$, the "output" fuzzy set $\nu_{\text{conclusion}}$ is computed as the composition of the fuzzy relation $\varrho_{\odot}$ and the fuzzy set $\mu_{A'}$. The composition of a fuzzy relation and a fuzzy set is defined as a generalization of the composition of an ordinary relation $R \subseteq X \times Y$ with an ordinary set $M \subseteq X$.

$$M \circ R \;=\; \{y \in Y \mid (\exists x \in X)(x \in M \text{ and } (x,y) \in R)\}$$

Valuating the existential quantifier in this formula, as usually, by the supremum and the conjunction by the operation $\sqcap \in \{\wedge, *\}$, we obtain the following definition of the composition of a fuzzy relation $\varrho \in L^{X\times Y}$ and a fuzzy set $\mu \in L^X$:

$$(\mu \circ_{\sqcap} \varrho)(y) \;=\; \bigvee_{x\in X} \{\mu(x) \sqcap \varrho(x,y)\} \tag{12}$$

for all $y \in Y$, where $\sqcap \in \{\wedge, *\}$ (cf., e.g., [5, 7, 17]).

Example 4.1. Let $L = [0,1]$ with the usual ordering. For $\sqcap = \wedge = \min$ and $\odot = \wedge$, (12) becomes

$$(\mu \circ \varrho)(y) \;=\; \bigvee_{x\in X} \min\{\mu(x), \mu_A(x), \mu_B(y)\}, \tag{13}$$

i.e.

$$\varrho(x,y) \;=\; \min\{\mu_A(x), \mu_B(y)\}. \tag{14}$$

This is the usual inference scheme for a single rule in fuzzy control, where in most cases μ is the characteristic function of a set with a single element representing the crisp input value $x_0 \in X$. In this case, (13) simplifies to

$$(\mu_{x_0} \circ \varrho)(y) \;=\; \min\{\mu_A(x_0), \mu_B(y)\}.$$

Another interesting example is $\sqcap = \wedge = \min$, $* = \wedge$, and $\odot = \rightarrow$. In this case $\odot$ is the Gödel implication, so that we have

$$\varrho(x, y) \;=\; \begin{cases} 1 & \text{if } \mu_A(x) \leq \mu_B(y), \\ \mu_B(y) & \text{otherwise.} \end{cases} \tag{15}$$

Both fuzzy relations (14) and (15) play an important role for the solution of fuzzy relation equations of the form

$$\mu_A \circ \varrho \;=\; \mu_B \tag{16}$$

with given fuzzy sets μ_A and μ_B and unknown fuzzy relation ϱ. For the usual sup–min composition, the operation $\circ$ is defined on the basis of $\sqcap = \wedge$. If the fuzzy relation Equation (16) has at least one solution ϱ, then (14) and (15) are also solutions (see, e.g., [7]). The greatest solution (15) is also very useful for systems of fuzzy relation equations, since, by taking the minimum over the solutions of the single equations, one obtains a solution of the system of equations when there exists a solution of the system at all.

Let us remark that it is not reasonable to consider all possible combinations of the choices for $\sqcap$ and $\odot$ in (12) with $\varrho = \varrho_\odot$ in Equation (11), as the following lemma shows.

Lemma 4.2. *Let $\mu_A \in L^X$ be a normal fuzzy set, i.e. there exists $x \in X$ such that $\mu_A(x) = \mathbb{1}$ holds, and let $\mu_B \in L^Y$. Define ϱ as in Equation* (11). *Then the equation*

$$(\mu_A \circ_\sqcap \varrho)(y) \;=\; \bigvee_{x \in X} \{\mu(x) \sqcap \varrho(x, y)\} \;=\; \mu_B(y) \tag{17}$$

is satisfied for all $y \in Y$ for the combinations $\odot = \rightarrow$ and $\sqcap = *$, $\odot = *$ and $\sqcap = *$, $\odot = *$ and $\sqcap = \wedge$, $\odot = \wedge$ and $\sqcap = *$, $\odot = \wedge$ and $\sqcap = \wedge$.

Proof.

(i) $\odot = \rightarrow$ and $\sqcap = *$.
Taking the normality of μ_A and Lemma 2.3(a) into account, we obtain:

$$\begin{aligned} (\mu_A \circ_\sqcap \varrho)(y) \;&=\; \bigvee_{x \in X} \{\mu_A(x) * (\mu_A(x) \rightarrow \mu_B(y))\} \\ &=\; \bigvee_{x \in X} \{\mu_A(x) \wedge \mu_B(y)\} \\ &=\; \mu_B(y). \end{aligned}$$

(ii) $\odot = *$ and $\sqcap = *$.
The normality of μ_A immediately yields

$$(\mu_A \circ_{\sqcap} \varrho)(y) \;=\; \bigvee_{x \in X} \{\mu_A(x) * (\mu_A(x) * \mu_B(y))\} \;=\; \mu_B(y).$$

(iii) $\odot = *$ and $\sqcap = \wedge$.
The normality of μ_A gives

$$(\mu_A \circ_{\sqcap} \varrho)(y) \;=\; \bigvee_{x \in X} \{\mu_A(x) \wedge (\mu_A(x) * \mu_B(y))\} \;=\; \mu_B(y).$$

(iv) $\odot = \wedge$ and $\sqcap = *$.
Again using the normality of μ_A we obtain:

$$(\mu_A \circ_{\sqcap} \varrho)(y) \;=\; \bigvee_{x \in X} \{\mu_A(x) * (\mu_A(x) \wedge \mu_B(y))\} \;=\; \mu_B(y).$$

(v) $\odot = \wedge$ and $\sqcap = \wedge$.
Finally, the normality of μ_A also ensures

$$(\mu_A \circ_{\sqcap} \varrho)(y) \;=\; \bigvee_{x \in X} \{\mu_A(x) \wedge (\mu_A(x) \wedge \mu_B(y))\} \;=\; \mu_B(y).$$

□

Example 4.3. For the missing combination $\odot = \rightarrow$ and $\sqcap = \wedge$ in Lemma 4.2 equation (17) does in general not hold. Let $L = [0, 1]$ be endowed with the usual ordering and the Łukasiewicz conjunction $*$. Consider $X = Y = \{a, b\}$, $\mu_A(a) = 1$, $\mu_A(b) = 0.9$, $\mu_B(a) = 1$, $\mu_B(b) = 0.8$. Then we obtain:

$$\begin{aligned}
(\mu_A \circ_{\sqcap} \varrho)(b) &= \max\{\mu_A(a) \wedge (\mu_A(a) \rightarrow \mu_B(b)), \mu_A(b) \wedge (\mu_A(b) \rightarrow \mu_B(b))\} \\
&= \max\{0.8, 0.9\} \\
&= 0.9 \\
&> \mu_B(b).
\end{aligned}$$

We are now prepared to formulate two interesting and important theorems about applying if–then rules and the indistinguishability inherent to the corresponding fuzzy sets.

Theorem 4.4. *Let $\mu, \mu_A \in L^X$, $\mu_B \in L^Y$. Furthermore, let E be a fuzzy equivalence relation on X such that μ_A is extensional w.r.t. E. Let $\varrho_\odot$ be defined as in Equation (11). Then for the combinations $\odot = \rightarrow$ and $\sqcap = *$, $\odot = *$ and $\sqcap = *$, $\odot = \wedge$ and $\sqcap = *$, the equation (cf. Equation (12))*

$$(\mu \circ_\sqcap \varrho_\odot) \;=\; (\hat{\mu} \circ_\sqcap \varrho_\odot)$$

is valid.

Proof. $\hat{\mu} \geq \mu$, together with the isotonicity of $\sqcap$, implies that

$$(\mu \circ_\sqcap \varrho_\odot) \;\leq\; (\hat{\mu} \circ_\sqcap \varrho_\odot)$$

for any choice of $\odot$ and $\sqcap$. Therefore we only have to prove the other inequality.

(i) $\odot = \rightarrow$ and $\sqcap = *$.

$$\begin{aligned}
(\hat{\mu} \circ_\sqcap \varrho_\odot)(y) &= \bigvee_{x \in X} \{\hat{\mu}(x) * (\mu_A(x) \rightarrow \mu_B(y))\} \\
&= \bigvee_{x,x' \in X} \Big\{(\mu(x') * E(x,x')) * (\mu_A(x) \rightarrow \mu_B(y))\Big\} \\
&\leq \bigvee_{x,x' \in X} \Big\{\mu(x') * E(x,x') * \Big((E(x,x') * \mu_A(x')) \rightarrow \mu_B(y)\Big)\Big\} \\
&= \bigvee_{x,x' \in X} \Big\{\mu(x') * E(x,x') * \Big(E(x,x') \rightarrow (\mu_A(x') \rightarrow \mu_B(y))\Big)\Big\} \\
&= \bigvee_{x,x' \in X} \Big\{\mu(x') * \Big(E(x,x') \wedge (\mu_A(x') \rightarrow \mu_B(y))\Big)\Big\} \\
&\leq \bigvee_{x' \in X} \Big\{\mu(x') * (\mu_A(x') \rightarrow \mu_B(y))\Big\} \\
&= (\mu \circ_\sqcap \varrho_\odot)(y).
\end{aligned}$$

In the second line we used the infinite distributive law, in the third the extensionality of μ_A together with Lemma 2.3(e), in the fourth Lemma 2.3(f), in the fifth Lemma 2.3(a), and in the sixth the isotonicity of $*$.

(ii) $\odot = *$ and $\sqcap = *$.
In accordance to the infinite distributive law and the extensionality of μ_A, we have that

$$\begin{aligned}(\hat{\mu} \circ_{\sqcap} \varrho_{\odot})(y) &= \bigvee_{x,x' \in X} \Big\{\mu(x') * E(x,x') * \mu_A(x) * \mu_B(y)\Big\} \\ &\leq \bigvee_{x' \in X} \Big\{\mu(x') * \mu_A(x') * \mu_B(y)\Big\} \\ &= (\mu \circ_{\sqcap} \varrho_{\odot})(y).\end{aligned}$$

(iii) $\odot = \wedge$ and $\sqcap = *$.

$$\begin{aligned}(\hat{\mu} \circ_{\sqcap} \varrho_{\odot})(y) &= \bigvee_{x,x' \in X} \Big\{\mu(x') * E(x,x') * (\mu_A(x) \wedge \mu_B(y))\Big\} \\ &\leq \bigvee_{x,x' \in X} \Big\{\mu(x') * \Big((E(x,x') * \mu_A(x)) \wedge (E(x,x') * \mu_B(y))\Big)\Big\} \\ &\leq \bigvee_{x' \in X} \Big\{\mu(x') * `(\mu_A(x') \wedge \mu_B(y))\Big\} \\ &= (\mu \circ_{\sqcap} \varrho_{\odot})(y).\end{aligned}$$

In the second line we applied Lemma 2.3(b). □

When we interpret Theorem 4.4 in the sense that the fuzzy sets μ_A and μ_B represent the linguistic terms A and B of an if–then rule of the form (9), then it states that for the mentioned combinations of operations for a given input μ the output fuzzy set $\mu \circ_{\sqcap} \varrho_{\odot}$ inferred by the rule does not change when we replace μ by its extensional hull. This means that the indistinguishability inherent to the fuzzy set μ_A cannot be avoided, even if the input fuzzy set μ stands for a crisp value. Note that if μ is a crisp set, the choice of the operation $\sqcap$ has no influence in Equation (12).

From Theorem 4.4 we derive also an answer to the question of fuzzy inputs. A fuzzified input does not change the outcome of a rule as long as the fuzzy set obtained by the fuzzification is contained in the extensional of the original crisp input value. From Theorem 4.4 we also learn that it does not make sense to measure more exactly than the indistinguishability admits.

The cases, that are covered by Theorem 4.4, include the most common and useful formalizations of if–then rules (see [2, 3]), namely the scheme $\odot = \wedge$ and $\sqcap = *$ with $* = \wedge$ as well as the scheme $\odot = \rightarrow$ and $\sqcap = *$ with $* = \wedge$.

The following example shows that Theorem 4.4 does not hold for the other combinations.

Example 4.5. For all counterexamples we assume $L = [0, 1]$ endowed with the usual ordering and the Łukasiewicz conjunction.

(i) $\odot = \rightarrow$ and $\sqcap = \wedge$.
Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, $\mu_A(x_1) = 1$, $\mu_A(x_2) = 0.5$, $\mu_B(y_1) = 0$, $\mu_B(y_2) = 1$, $E(x_1, x_2) = 0.5$, $\mu(x_1) = 1$, $\mu(x_2) = 0$. Then we have $(\hat{\mu} \circ_\sqcap \varrho_\odot)(y_1) = 0.5 > 0 = (\mu \circ_\sqcap \varrho_\odot)(y_1)$.

(ii) $\odot = *$ and $\sqcap = \wedge$.
Let $X = \{0, 0.5, 1\}$, $Y = \{y_1, y_2\}$, $\mu_A(x) = x$, $\mu_B(y_1) = 1$, $\mu_B(y_2) = 0$, $E(x, x') = 1 - \min\{|x - x'|, 1\}$, $\mu(0) = 1$, $\mu(0.5) = \mu(1) = 0$. Then we have $(\hat{\mu} \circ_\sqcap \varrho_\odot)(y_1) = 0.5 > 0 = (\mu \circ_\sqcap \varrho_\odot)(y_1)$.

(iii) $\odot = \wedge$ and $\sqcap = \wedge$.
Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$, $\mu_A(x_1) = 1$, $\mu_A(x_2) = 0.5$, $\mu_A(x_3) = 0$, $\mu_B(y_1) = 0$, $\mu_B(y_2) = 1$, $E(x_1, x) = \mu_A(x)$, $x \in X$. $E(x_2, x_3) = 0.5$, $\mu(x_3) = 1$, $\mu(x_1) = \mu(x_2) = 0$. Then we have $(\hat{\mu} \circ_\sqcap \varrho_\odot)(y_1) = 0.5 > 0 = (\mu \circ_\sqcap \varrho_\odot)(y_1)$.

Theorem 4.4 shows that the indistinguishability induced by the fuzzy set representing the linguistic in the premise of the rule cannot be overcome. The same holds for the output of the rule, i.e. the output can never be more precise than the indistinguishability induced by the fuzzy set modelling the linguistic term in the conclusion of the rule.

Theorem 4.6. *Let $\mu, \mu_A \in L^X$, $\mu_B \in L^Y$. Furthermore, let F be a fuzzy equivalence relation on Y such that μ_B is extensional w.r.t. F. Let $\varrho_\odot$ be defined as in Equation* (11)*. Then for the combinations $\odot = \rightarrow$ and $\sqcap = *$, $\odot = *$ and $\sqcap = *$, $\odot = \wedge$ and $\sqcap = *$, the fuzzy set $(\mu \circ_\sqcap \varrho_\odot)$ (cf. Equation* (12)*) is extensional w.r.t. F.*

Proof.

(i) $\odot = \rightarrow$ and $\sqcap = *$.
Taking Lemma 2.3(g), (h) and the extensionality of μ_B into account, we obtain:

$$\begin{aligned}
(\mu \circ_\sqcap \varrho_\odot)(y) * F(y,y') &= \bigvee_{x\in X} \Big\{\mu(x) * (\mu_A(x) \rightarrow \mu_B(y)) * F(y,y')\Big\} \\
&\leq \bigvee_{x\in X} \Big\{\mu(x) * \Big(\mu_A(x) \rightarrow (\mu_B(y) * F(y,y'))\Big)\Big\} \\
&\leq \bigvee_{x\in X} \Big\{\mu(x) * (\mu_A(x) \rightarrow \mu_B(y'))\Big\} \\
&= (\mu \circ_\sqcap \varrho_\odot)(y').
\end{aligned}$$

(ii) $\odot = *$ and $\sqcap = *$.

$$\begin{aligned}
(\mu \circ_\sqcap \varrho_\odot)(y) * F(y,y') &= \bigvee_{x\in X} \Big\{\mu(x) * \mu_A(x) * \mu_B(y) * F(y,y')\Big\} \\
&\leq \bigvee_{x\in X} \Big\{\mu(x) * \mu_A(x) * \mu_B(y')\Big\} \\
&= (\mu \circ_\sqcap \varrho_\odot)(y').
\end{aligned}$$

(iii) $\odot = \wedge$ and $\sqcap = *$.
Using Lemma 2.3(b), we get

$$\begin{aligned}
(\mu \circ_\sqcap \varrho_\odot)(y) * F(y,y') &= \bigvee_{x\in X} \Big\{\mu(x) * ((\mu_A(x) \wedge \mu_B(y)) * F(y,y')\Big\} \\
&\leq \bigvee_{x\in X} \Big\{\mu(x) * \\
&\qquad \Big((\mu_A(x) * F(y,y')) \wedge (\mu_B(y) * F(y,y'))\Big)\Big\} \\
&\leq \bigvee_{x\in X} \Big\{\mu(x) * \mu_A(x) * \mu_B(y')\Big\} \\
&= (\mu \circ_\sqcap \varrho_\odot)(y').
\end{aligned}$$

□

In Theorems 4.4 and 4.6 we have only considered a single pair of fuzzy sets μ_A and μ_B that model the linguistic term in the premise, respectively the conclusion of a single if–then rule of the form (9). However, our results can easily be extended to the more realistic case of a set of if–then rules.

Consider the set of rules

$$\text{If } \xi \text{ is } A_i, \text{ then } \eta \text{ is } B_{,} \qquad (i \in I),$$

where the linguistic terms A_i and B_i are modelled by the fuzzy set $\mu_{A_i} \in L^X$ and $\mu_{B_i} \in L^Y$.

Given an "input fuzzy set" $\mu \in L^X$, the output of this set of rules is computed in the following way: as a first step, a combination of the operations $\odot$ and $\sqcap$ is chosen. Let us assume that one of the three cases in Theorems 4.4 and 4.6 is considered. Then for each single rule, the corresponding fuzzy relation $\varrho_i(x,y) = \mu_{A_i} \odot \mu_{B_i}$ and also on this basis, the corresponding output fuzzy set $\mu \circ_\sqcap \varrho_i$ is computed. Finally, these outputs are aggregated usually either by taking the infimum or supremum, i.e. either

$$\bigwedge_{i \in I} (\mu \circ_\sqcap \varrho_i) \tag{18}$$

or

$$\bigvee_{i \in I} (\mu \circ_\sqcap \varrho_i). \tag{19}$$

Generalizing Theorem 4.4 to a set of rules, the result remains the same since in accordance to Theorem 4.4, replacing μ by its extensional hull, does not change the fuzzy set $\mu \circ_\sqcap \varrho_i$ so that neither (18) nor (19) is affected.

The result of Theorem 4.6 is also valid for a set of rules, i.e. the output fuzzy set is extensional, since, due to Theorem 3.2, infima and suprema maintain extensionality.

The only important thing is that one has to consider fuzzy equivalence relations E and F on X, respectively Y, such that all fuzzy sets μ_{A_i} (respectively μ_{B_i}) are extensional with respect to E (respectively F). Of course, the most interesting fuzzy equivalence relations are the coarsest ones since they characterize the indistinguishability inherent to the given fuzzy sets and yield the greatest extensional hulls. The coarsest fuzzy equivalence relation making a given collection of fuzzy sets extensional, is described in Theorem 3.1.

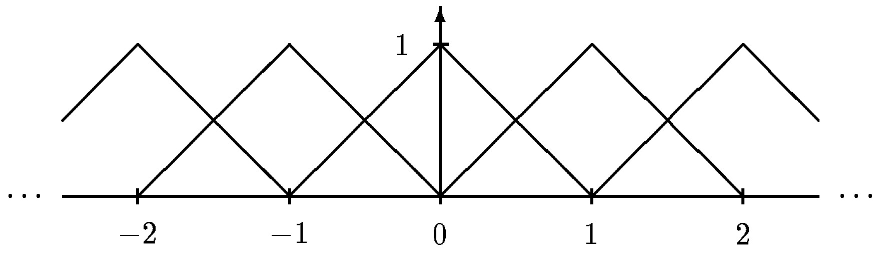

Figure 1: A typical fuzzy partition.

Example 4.7. Let $L = [0,1]$. We consider a typical "fuzzy partition" of isosceles triangles on the real numbers, i.e. the set of fuzzy sets $\{\mu_i \mid i \in \mathbb{Z}\}$ where $\mu_i(x) = 1 - \min\{|x-i|, 1\}$ (see Figure 1).

For the most often applied approximate reasoning schemes for if–then rules described in Example 4.1, we have to choose $* = \wedge$. Therefore the fuzzy equivalence relation induced by these fuzzy sets is

$$\begin{aligned} E(x,y) &= \bigwedge_{i \in I} (\mu_i(x) \leftrightarrow \mu_i(y)) \\ &= \begin{cases} 1 & \text{if } x = y, \\ \min\{i+1-x, x-i, i+1-y, y-i\} & \text{if } i \in \mathbb{Z} \text{ and} \\ & i < x, y < i+1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Figure 2 illustrates the extensional hulls of the crisp values 0.5 and 1.75 with respect to this fuzzy equivalence relation, i.e. the fuzzy sets $\mu_{0.5}(x) = E(x, 0.5)$ and $\mu_{1.75}(x) = E(x, 1.75)$. The greatest indistinguishability is reached at the intermediate points $(z + 0.5)$ $(z \in \mathbb{Z})$ whereas the extensional hull of the points $z \in \mathbb{Z}$ remains crisp leading to optimal accuracy.

It should be emphasized that the choice of $* = \wedge$ leads to the greatest distinguishability since the minimum is the greatest t–norm. For any other left continuous t–norm, the implication obtained by residuation and therefore also the corresponding biimplication, yields greater values than the implication, respectively biimplication, induced by the minimum so that the fuzzy equivalence relation leads also to greater values (higher indistinguishability).

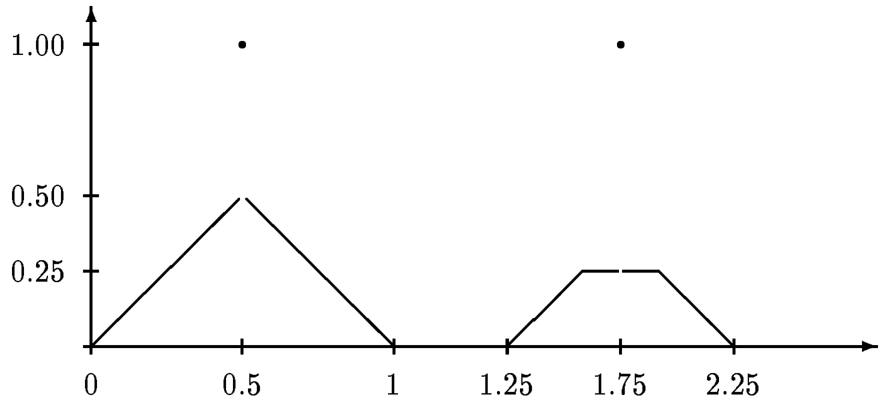

Figure 2: The extensional hulls of the crisp values 0.5 and 1.75.

The if–then rules considered had only one variable in the premise and the conclusion. Nevertheless, these variables may also be vectors so that we have included the cases of multiple inputs and outputs as well. In such cases the corresponding fuzzy equivalence relations are defined on multi–dimensional product spaces. In some cases it is however possible to compute a lower approximation of a fuzzy equivalence relation in a product space by fuzzy equivalence relations in one–dimensional spaces.

Let us assume the if–then rules are of the form

"If ξ_1 is $A_i^{(1)}$ and ... and ξ_n is $A_i^{(n)}$, then η is B_i,"

with $X_1, \ldots, X_n$ as the underlying domains for the variables $\xi_1, \ldots, \xi_n$. The linguistic terms $A_i^{(k)}$ are modelled by the fuzzy sets $\mu_i^{(k)} \in L^{X_k}$. We are interested in the fuzzy equivalence relation induced by these fuzzy sets on the product space $X_1 \times \ldots \times X_n$. Therefore it is necessary to know how the *and*–expression appearing in the if–then rules is interpreted. Let us assume that, as usually, the rule is evaluated by using $\wedge$ for the *and*–expression. Then we could rewrite the rules in the form

"If $(\xi_1, \ldots, \xi_n)$ is $(A_i^{(1)}, \ldots, A_i^{(n)})$, then η is B_i," $(i \in I)$, where the linguistic term $(A_i^{(1)}, \ldots, A_i^{(n)})$ is represented by the fuzzy set

$$\left(\bigwedge_{k=1}^{n} \mu_i^{(k)} \right) \in L^{X_1 \times \ldots \times X_n}.$$

Thus the corresponding fuzzy equivalence relation on the product space $X_1 \times \ldots \times X_n$ based on formula (5) is

$$E((x_1, \ldots, x_n), (x_1', \ldots, x_n')) = \bigwedge_{i \in I} \left(\Big(\bigwedge_{k=1}^{n} \mu_i^{(k)}(x_k) \Big) \leftrightarrow \Big(\bigwedge_{k=1}^{n} \mu_i^{(k)}(x_k') \Big) \right). \tag{20}$$

The following theorem shows that a lower approximation for this fuzzy equivalence relation can be derived from the fuzzy equivalence relations in the one–dimensional spaces.

Theorem 4.8. *Let $\mu_i^{(k)} \in L^{X_k}$, ($i \in I$, $k \in \{1, \ldots, n\}$). Let E be the fuzzy equivalence relation defined in* (20). *Define*

$$E_k(x, y) = \bigwedge_{i \in I} \left(\mu_i^{(k)}(x) \leftrightarrow \mu_i^{(k)}(y) \right)$$

for all $x, y \in X_k$. Furthermore, let

$$\tilde{E}((x_1, \ldots, x_n), (x_1', \ldots, x_n')) = \bigwedge_{k=1}^{n} E_k(x_k, x_k').$$

Then $\tilde{E}$ is a fuzzy equivalence relation on $X_1 \times \ldots \times X_n$ and $\tilde{E} \le E$.

Proof. The fuzzy sets $\mu_i^{(k)} \in L^{X_k}$ can be interpreted as fuzzy sets $\tilde{\mu}_i^{(k)} \in L^{X_1 \times \ldots \times X_n}$ by defining

$$\tilde{\mu}_i^{(k)}(x_1, \ldots, x_n) = \mu_i^{(k)}(x_k).$$

Thus $\tilde{E}$ can be rewritten in the form

$$\tilde{E}((x_1, \ldots, x_n), (x_1', \ldots, x_n')) = \bigwedge_{i \in I} \bigwedge_{k=1}^{n} \left(\tilde{\mu}_i^{(k)}(x_1, \ldots, x_n) \leftrightarrow \tilde{\mu}_i^{(k)}(x_1', \ldots, x_n') \right)$$

and is therefore, by Theorem 3.1, a fuzzy equivalence relation on $X_1 \times \ldots \times X_n$. Lemma 2.3(j) leads immediately to $\tilde{E} \le E$. □

Let us finally remark that we did not consider the defuzzification problem here. Although defuzzification aims at "deleting" the indistinguishability or imprecision inherent to a fuzzy set, our result that the indistinguishability inherent to a given set of fuzzy sets cannot be overcome by a suitable defuzzification strategy, since defuzzification is usually applied to the final output fuzzy set of an approximate reasoning scheme. But this fuzzy set remains the same when the input fuzzy set is replaced by its extensional hull, so that defuzzification will also yield the same crisp output value.

5 Conclusions

We have shown that fuzzy equivalence relations are a useful model to describe the indistinguishability inherent to fuzzy sets. Approximate reasoning schemes which are used in fuzzy control and other fields cannot avoid or overcome this indistinguishability. Thus fuzzy equivalence relations characterize a kind of granularity of the model. This information can be used to determine a limit for the degree of precision in which inputs should be measured, since a higher accuracy than the indistinguishability inherent to the fuzzy sets or fuzzy equivalence relations does not influence the resulting output of a fuzzy system. The fuzzy equivalence relations also describe how accurate or precise the outputs of a fuzzy system can principally be.

Taking these ideas into account, in future works, it would be interesting to develop the notion of an indistinguishability measure for fuzzy equivalence relations analogous to entropy or information measures of probability theory.

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