A Formal Study of Linearity Axioms for Fuzzy Orderings

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Abstract

This contribution is concerned with a detailed investigation of linearity axioms for fuzzy orderings. Different existing concepts are evaluated with respect to three fundamental correspondences from the classical case—linearizability of partial orderings, intersection representation, and one-to-one correspondence between linearity and maximality. As a main result, we obtain that it is virtually impossible to simultaneously preserve all these three properties in the fuzzy case. If we do not require a one-to-one correspondence between linearity and maximality, however, we obtain that an implication-based definition appears to constitute a sound compromise, in particular, if Łukasiewicz-type logics are considered.

Key words: completeness, fuzzy ordering, fuzzy preference modeling, fuzzy relation, linearity, Szpilrajn theorem.

1 Introduction

Orderings are fundamental concepts in mathematics, among which linear orderings play an outstanding role [29]. Beside the context of orderings, in a more general setting, the linearity property also has a great importance in modeling of preferences by relational constructs, since it corresponds to the important property of full comparability (often called *completeness*) or, in other words, absence of incomparability.

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Fuzzy relations have been introduced in order to provide more flexible models for expressing relationships [13, 15, 23, 25, 26, 35]. The appropriate definition of completeness/linearity, however, is by far not as straightforward as in the classical Boolean case. Several different approaches appear in literature; a systematic formal study with respect to deep logical and algebraic properties, however, has not yet been conducted so far.

The aim of this paper is to investigate three existing definitions of completeness of fuzzy relations in detail. For that purpose, we consider the most fundamental relations for which completeness plays a role—fuzzy orderings—and evaluate the different notions of linearity with respect to fundamental deep results that hold in the crisp case. The final goal is to gain deeper insight into the principles of existing linearity axioms in order to have clear arguments pro and contra their use, not only in connection with fuzzy orderings, but also in more general settings in fuzzy preference modeling.

2 Fundamental Properties of Crisp Orderings

In order to clear up notation, let us briefly recall classical orderings (let us synonymously use the term *crisp* for Boolean, classical, or non-fuzzy). Throughout the whole paper, assume that the symbol X denotes an arbitrary non-empty set.

Definition 1 A binary relation $\lesssim$ on the set X (i.e. a two-place predicate on the product set $X \times X$) is called *(partial) ordering* if and only if it fulfills the following three axioms (for all $x,y,z \in X$):

Reflexivity:	$x \lesssim x$
Antisymmetry:	$(x \lesssim y \wedge y \lesssim x) \;\Rightarrow\; x = y$
Transitivity:	$(x \lesssim y \wedge y \lesssim z) \;\Rightarrow\; x \lesssim z$

Definition 2 A binary relation $\Diamond$ on X is called *complete* if and only if

$$x \Diamond y \vee y \Diamond x \tag{2.1}$$

holds for any pair $x,y \in X$. An ordering fulfilling completeness is called *linear ordering*.

Since this will be important in the following, let us briefly note that (2.1) is equivalent to

$$x \not\Diamond y \;\Rightarrow\; y \Diamond x. \tag{2.2}$$

Completeness is just a simple axiomatization of a property which has a much deeper meaning in logical and algebraic terms. In particular, there are three essential aspects of relationship between arbitrary orderings and linear orderings:

[SZP] Any partial ordering can be linearized *(Szpilrajn's Theorem)* [31]: For any partial ordering $\lesssim$, there exists a linear ordering $\preceq$ which extends $\lesssim$ in the sense that, for all $x,y \in X$,

$$x \lesssim y \;\Rightarrow\; x \preceq y. \tag{2.3}$$

[INT] Any partial ordering can be represented as an intersection of linear orderings [11]: For any partial ordering $\lesssim$, there exists a family of linear orderings $(\preceq_i)_{i\in I}$ such that $\lesssim$ can be represented as (for all $x,y \in X$)

$$x \lesssim y \;\Leftrightarrow\; \bigwedge_{i\in I} x \preceq_i y.$$

[MAX] There is a one-to-one correspondence between linearity and maximality: An ordering $\lesssim$ is linear if and only if there exists no non-trivial extension, i.e. the only ordering $\preceq$ fulfilling (2.3) is $\lesssim$ itself.

These three fundamentally important correspondences will serve as the criteria for evaluating fuzzy linearity/completeness axioms in this paper.

3 Fuzzy Orderings

Binary fuzzy relations were proposed to provide additional freedom for expressing complex preferences that can rarely be modeled in the rigid setting of bivalent logic [13,15,23,25,26,35]. This is accomplished—as usual in fuzzy set theory—by allowing intermediate degrees of relationship. This paper assumes that the domain of truth values is the common unit interval $[0,1]$.

Definition 3 Given a non-empty set X, a mapping $R : X^2 \to [0,1]$ is called *binary fuzzy relation* on X.

The considerations in this paper use triangular norms and related operations to model logical operators and connectives [22, 30].

Definition 4 A *triangular norm* (*t-norm* for short) is an associative, commutative, and non-decreasing binary operation on the unit interval (i.e. a $[0,1]^2 \to [0,1]$ mapping) which has 1 as neutral element.

A well-studied class of fuzzy relations that will also be of central importance for this paper are so-called fuzzy equivalence relations. [1] They are nowadays widely accepted as proper fuzzifications of classical equivalence relations [5,20,21,25,26, 32,33,35].

[1] Note that various diverging names for this class appear in literature, like similarity relations, indistinguishability operators, equality relations, etc.

Definition 5 A binary fuzzy relation E on X is called *fuzzy equivalence relation* with respect to T, for brevity *T-equivalence*, if and only if the following three axioms are fulfilled for all $x, y, z \in X$:

Reflexivity: $E(x,x) = 1$
Symmetry: $E(x,y) = E(y,x)$
T-transitivity: $T\big(E(x,y), E(y,z)\big) \leq E(x,z)$

Fuzzy relations only fulfilling reflexivity and T-transitivity are called *preorderings* with respect to t-norm T, for short, *T-preordering*.

This paper addresses the so far most general notion of fuzzy orderings that—in contrast to earlier approaches—takes an underlying concept of equality/equivalence into account [2–4, 19]. This equality/equivalence is modeled by a fuzzy equivalence relation.

Definition 6 Let $L : X^2 \to [0,1]$ be a T-transitive fuzzy relation. L is called *fuzzy ordering* with respect to T and a T-equivalence E, for brevity *T-E-ordering*, if and only if it additionally fulfills the following two axioms for all $x, y \in X$:

E-Reflexivity: $E(x,y) \leq L(x,y)$
T-E-antisymmetry: $T\big(L(x,y), L(y,x)\big) \leq E(x,y)$

Before the general concept above was introduced, fuzzy orderings were rather commonly understood as T-preorderings that additionally fulfill T-antisymmetry [15, 35], i.e., for all $x, y \in [0,1]$,

$$x \neq y \;\Rightarrow\; T\big(L(x,y), L(y,x)\big) = 0.$$

In order to avoid misunderstandings, let us call this class of fuzzy orderings T-orderings. As easy to observe, Definition 6 still accommodates T-orderings if we define E to be the crisp equality. It turned out that basing fuzzy orderings on the crisp equality is too restrictive and practically not feasible. A detailed argumentation is elaborated in [2, 3].

Already in Zadeh's very first paper on fuzzy orderings [35], the fundamental property [SZP] is addressed. If the minimum t-norm is considered for modeling transitivity and antisymmetry (as usual in Zadeh's early works), [SZP] is guaranteed to be satisfied. The proof of this result is simple by using the classical Szpilrajn theorem [31]. A straightforward generalization of this theorem to t-norms without zero divisors was later proved by Gottwald [15]. Although these results seem encouraging at first glance, they do not provide much insight. Nonchalantly speaking, T-orderings, in particular if T does not have zero divisors, are almost crisp concepts. Consequently, [SZP] follows instantly. However, this result relies on the crispness of the concepts under investigation and is by no means applicable if we admit a non-trivial concept of fuzzy equivalence à la Definition 6.

A first serious attempt to investigate [SZP] and [INT] for fuzzy orderings in the sense of Definition 6 was made by Höhle and Blanchard [19]. This paper provides a specific definition of linearity/completeness that has neither become common nor widely known, as it unfortunately remained unknown to the vast majority of the fuzzy set community.

The given paper puts the dispersed attempts and approaches existing in literature into a common perspective. It considers three major approaches to modeling linearity/completeness—two common in fuzzy preference modeling and the one due to Höhle and Blanchard. All three concepts are checked against the three fundamental properties. In any case, we say that a given concept of linearity/completeness fulfills one of the three fundamental properties if and only if the property is satisfied for all domains X and all T-equivalences E—as a restriction to specific domains or T-equivalences would contradict the generic nature of the fundamental properties in the crisp case. The choice of the logical operators and connectives is crucial for the specific logical framework under investigation. Where possible, characterizations are provided which conditions the logical operators and connectives have to satisfy in order to guarantee that a concept of linearity/completeness fulfills a particular fundamental property.

4 Preliminaries: Fuzzy Logical Connectives

This paper makes fundamental use of triangular norms and related operations. In order to make this paper as self-contained as possible, we briefly provide the reader with the most important basics in a consistent notation. For details, the reader is referred to the literature, e.g [13, 22].

We first give a brief overview of specific properties and important classes of t-norms that will be essential throughout this paper.

Definition 7 Special properties and classes of triangular norms:

(1) A t-norm T is said to have *zero divisors* if and only if there exists a pair $(x,y) \in (0,1)^2$ such that $T(x,y) = 0$ holds.
(2) A t-norm T is called *Archimedean* if and only if, for all pairs $(x,y) \in (0,1)^2$, there is an $n \in \mathbb{N}$ such that

$$T(\overbrace{x,\ldots,x}^{n \text{ times}}) < y$$

(3) A t-norm T is called *left-continuous* if the following holds for all $x \in [0,1]$ and all families $(y_i)_{i\in I} \in [0,1]^I$:

$$T\left(x, \sup_{i\in I} y_i\right) = \sup_{i\in I} T\left(x, y_i\right)$$

(4) A t-norm T is called *strictly monotone* if and only if $y < z$ always implies $T(x,y) < T(x,z)$ (for all $x,y,z \in [0,1]$).
(5) A strictly monotone and continuous t-norm is called *strict*.
(6) A t-norm T is called *nilpotent* if it is continuous and if, for all pairs $(x,y) \in (0,1)^2$, there is an $n \in \mathbb{N}$ such that

$$T(\overbrace{x,\ldots,x}^{n \text{ times}}) = 0$$

Theorem 8 [22,24,30] *A function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm if and only if there exists a continuous, strictly decreasing function $\varphi : [0,1] \to [0,\infty]$ with $\varphi(1) = 0$ called* additive generator *such that, for all $x,y \in [0,1]$, the following holds:*

$$T(x,y) = \varphi^{-1}\Big(\min\big(\varphi(x) + \varphi(y), \varphi(0)\big)\Big) \tag{4.1}$$

The generator φ is uniquely determined up to a positive multiplicative constant.

Corollary 9 [13,22] *A continuous Archimedean t-norm T is either strict or nilpotent. T is nilpotent if and only if $\varphi(0) < \infty$ holds (for some additive generator φ), otherwise T is strict.*

The following four operations are triangular norms, usually called minimum t-norm, product t-norm, Łukasiewicz t-norm, and nilpotent minimum, respectively:

$$\begin{aligned}
T_{\mathbf{M}}(x,y) &= \min(x,y) \\
T_{\mathbf{P}}(x,y) &= x \cdot y \\
T_{\mathbf{L}}(x,y) &= \max(x+y-1,0) \\
T_{\mathbf{nM}}(x,y) &= \begin{cases} \min(x,y) & \text{if } x+y>1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

All four operations are left-continuous, while the first three operations are even continuous. Only $T_{\mathbf{L}}$ and $T_{\mathbf{nM}}$ do have zero divisors. $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ are Archimedean, where $T_{\mathbf{P}}$ is strict and $T_{\mathbf{L}}$ is nilpotent. Note that $T_{\mathbf{nM}}$, although the name would suggest this, is *not nilpotent*. Moreover, it worth to mention that $T_{\mathbf{M}}$ is the unique largest t-norm.

Theorem 10 [22, 30] *Let $(T_i)_{i\in I}$ be a family of t-norms and let $\big((a_i,e_i)\big)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then the following function $T : [0,1]^2 \to [0,1]$ is a t-norm:*

$$T(x,y) = \begin{cases} a_i + (e_i - a_i)\cdot T_i\!\left(\frac{x-a_i}{e_i-a_i}, \frac{y-a_i}{e_i-a_i}\right) & \textit{if } (x,y) \in [a_i,e_i]^2 \\ \min(x,y) & \textit{otherwise} \end{cases}$$

The t-norm T is called the ordinal sum *of the* summands $\langle a_i, e_i, T_i\rangle$*, and we write $T = (\langle a_i,e_i,T_i\rangle)_{i\in I}$. Moreover, a t-norm is continuous if and only if it is an ordinal sum with continuous Archimedean summands.*

We use triangular conorms as generalized models of disjunction.

Definition 11 A *triangular conorm* (*t-conorm* for short) is an associative, commutative, and non-decreasing binary operation on the unit interval which has 0 as neutral element.

Usually, only t-norms and t-conorms are considered together which are linked by means of a generalized de Morgan law. In order to be able to define this, let us briefly recall generalized negations.

Definition 12 A non-increasing function $N : [0,1] \to [0,1]$ fulfilling the boundary conditions $N(0) = 1$ and $N(1) = 0$ is called *negation*.

Definition 13 A negation is called *strict* if and only if it is strictly decreasing and continuous. A strict negation N is called *strong* or *involutive* if and only if it is self-inverse, i.e., for all $x \in [0,1]$,

$$N\big(N(x)\big) = x.$$

Theorem 14 [13,27] *A negation N is strong if and only if there exists an automorphism* $\varphi : [0,1] \to [0,1]$ *such that N can be represented as follows (for all* $x \in [0,1]$*):*

$$N(x) = \varphi^{-1}\big(1 - \varphi(x)\big) \tag{4.2}$$

Definition 15 A triple (T,S,N), where T is a t-norm, S is a t-conorm, and N is a strong negation, is called *de Morgan triple* if and only if the de Morgan law

$$S(x,y) = N\big(T(N(x),N(y))\big)$$

is fulfilled for all $x,y \in [0,1]$. A de Morgan triple (T,S,N) is called *Łukasiewicz triple* if T is nilpotent.

Several investigations have shown [15–17] that the most meaningful concepts of fuzzy implications in logical terms are so-called residual implications. Since this notion will play a central role in our further investigations, we briefly recall the basic definitions and properties.

Definition 16 *For a left-continuous t-norm* T*, the residual implication* $\vec{T}$ *is defined as (*$x,y \in [0,1]$*)*

$$\vec{T}(x,y) = \sup\{u \in [0,1] \mid T(u,x) \le y\}.$$

Lemma 17 [15–17,22] *Provided that* T *is left-continuous, the following holds for all* $x,y,z \in [0,1]$*:*

(1) $T(x,y) \le z \;\Leftrightarrow\; x \le \vec{T}(y,z)$
(2) $x \le y \;\Leftrightarrow\; \vec{T}(x,y) = 1$
(3) $T\big(\vec{T}(x,y), \vec{T}(y,z)\big) \le \vec{T}(x,z)$

(4) $\vec{T}(1,y) = y$
(5) $T\big(x, \vec{T}(x,y)\big) \leq y$

Furthermore, $\vec{T}$ *is non-increasing and left-continuous in the first argument and non-decreasing and right-continuous in the second argument.*

Theorem 18 [13, 22] *Consider a continuous Archimedean t-norm* T*. Then its residuum can be represented as*

$$\vec{T}(x,y) = \varphi^{-1}\big(\max(\varphi(y) - \varphi(x), 0)\big), \tag{4.3}$$

where φ *denotes an arbitrary additive generator of* T*.*

The residual implication also determines a negation in a straightforward way.

Definition 19 The *negation* corresponding to a left-continuous t-norm T is defined as

$$N_T(x) = \vec{T}(x,0).$$

Lemma 20 [13] *For any left-continuous t-norm* T*,* N_T *is a negation. If* T *is additionally nilpotent,* N_T *is a strong negation. In this case,* N_T *can be represented as in* (4.2) *with* φ *being the additive generator that fulfills* $\varphi(0) = 1$.

5 Extensions and the Role of Left-Continuity

All three properties [SZP], [INT], and [MAX] consider extensions of a given fuzzy ordering. This section is devoted to basic definitions and properties that will be essential in the following.

Definition 21 Consider two T-E-orderings L_1 and L_2. We say that L_1 *extends* L_2 if and only if, for all $x,y \in X$, $L_2(x,y) \leq L_1(x,y)$ holds. For brevity we denote this $L_2 \subseteq L_1$. We call L_1 a *non-trivial extension of* L_2 if there exists at least one pair $(x,y) \in X^2$ for which $L_2(x,y) < L_1(x,y)$ holds, for brevity $L_2 \subset L_1$.

It is obvious that $\subseteq$ as defined above is a partial ordering on the set $[0,1]^{X\times X}$, i.e. it is nothing else but the Cartesian product of the natural linear ordering on the unit interval with respect to the index set $X \times X = X^2$.

Definition 22 We denote the up-set, the set of elements larger than or equal to (i.e. extending) a given T-E-ordering L, with

$$\text{ext}(L) = \{L' \mid L' \text{ is a } T\text{-}E\text{-ordering and } L \subseteq L'\}.$$

A T-E-ordering L is called *maximal* if and only if it does not have a non-trivial extension, equivalently, $\text{ext}(L) = \{L\}$.

As the next theorem demonstrates, the applicability of Zorn's Lemma in the context of extensions is strictly dependent on the left-continuity of the underlying t-norm.

Theorem 23 *Consider a T-E-ordering L. If T is left-continuous, the set* $\text{ext}(L)$ *has at least one maximal element.*

Proof. We consider an arbitrary linearly ordered sequence in $\text{ext}(L)$, i.e. a family $(L_i)_{i\in I}$ such that

(1) the index set I is linearly ordered,
(2) for all $i \in I$, $L_i \in \text{ext}(L)$,
(3) $L_i \subseteq L_j$ whenever $i \leq j$.

Now we define (for all $x, y \in X$)

$$\tilde{L}(x,y) = \sup_{i\in I} L_i(x,y).$$

E-reflexivity of $\tilde{L}$ is trivial to prove. For proving T-E-antisymmetry, we have to take left-continuity into account:

$$\begin{aligned} T\big(\tilde{L}(x,y), \tilde{L}(y,x)\big) &= T\big(\sup_{i\in I} L_i(x,y), \sup_{j\in I} L_j(y,x)\big) \\ &= \sup_{i\in I}\sup_{j\in I} T\big(L_i(x,y), L_j(y,x)\big) = (*) \end{aligned}$$

Since the family $(L_i)_{i\in I}$ is linearly ordered, the equality

$$(*) = \sup_{i\in I} T\big(L_i(x,y), L_i(y,x)\big)$$

holds, and T-antisymmetry follows from the T-E-antisymmetry of every L_i. A similar argumentation can be applied to prove T-transitivity:

$$\begin{aligned} T\big(\tilde{L}(x,y), \tilde{L}(y,z)\big) &= T\big(\sup_{i\in I} L_i(x,y), \sup_{j\in I} L_j(y,z)\big) \\ &= \sup_{i\in I}\sup_{j\in I} T\big(L_i(x,y), L_j(y,z)\big) \\ &= \sup_{i\in I} T\big(L_i(x,y), L_i(y,z)\big) \\ &\leq \sup_{i\in I} L_i(x,z) \\ &= \tilde{L}(x,z) \end{aligned}$$

Hence, we have shown constructively that any linearly ordered sequence in $\text{ext}(L)$ has a supremum in $\text{ext}(L)$. By Zorn's Lemma, therefore, the existence of a maximal element in $\text{ext}(L)$ is guaranteed. □

Now we turn to the opposite questions, how severe the difficulties are that arise if left-continuity is not satisfied.

Proposition 24 *Provided that the set X has at least two elements and that T is not left-continuous, there exists a T-equivalence E and a linearly ordered sequence of T-E-orderings which does not have a supremum in the set of T-E-orderings on X.*

Proof. Assume that a t-norm T is not left-continuous, i.e. there exists an $\alpha \in (0,1)$ and an ascending sequence $(\beta_n)_{n\in\mathbb{N}}$ such that

$$T\left(\alpha, \sup_{n\in\mathbb{N}} \beta_n\right) \neq \sup_{n\in\mathbb{N}} T(\alpha, \beta_n). \tag{5.1}$$

Since

$$\sup_{n\in\mathbb{N}} T(\alpha, \beta_n) \leq T\left(\alpha, \sup_{n\in\mathbb{N}} \beta_n\right)$$

always holds due to the monotonicity of T, (5.1) implies

$$\sup_{n\in\mathbb{N}} T(\alpha, \beta_n) < T\left(\alpha, \sup_{n\in\mathbb{N}} \beta_n\right) \leq \min\left(\alpha, \sup_{n\in\mathbb{N}} \beta_n\right) \tag{5.2}$$

Therefore, with the notation

$$\gamma = \sup_{n\in\mathbb{N}} T(\alpha, \beta_n)$$

we obtain that $\gamma < \alpha$ and that we can choose an n_0 such that, for all $n \geq n_0$, $\beta_n > \gamma$ holds. Without loss of generality, assume that $\beta_n > \gamma$ holds for all $n \in \mathbb{N}$, otherwise a sub-sequence can be considered.

Now let us consider an arbitrary linear ordering $\preceq$ of the domain X. Such a linear ordering always exists due to the classical Szpilrajn theorem. We define the following fuzzy relations on X (for $n \in \mathbb{N}$):

$$L_n(x,y) = \begin{cases} 1 & \text{if } x = y \\ \alpha & \text{if } x \prec y \\ \beta_n & \text{if } x \succ y \end{cases} \qquad E(x,y) = \begin{cases} 1 & \text{if } x = y \\ \gamma & \text{otherwise} \end{cases}$$

It is easy to see that E is reflexive, symmetric, and also T-transitive, therefore, a T-equivalence. Moreover, all L_n are trivially E-reflexive (as $\gamma < \alpha$ and $\gamma < \beta_n$ for all $n \in \mathbb{N}$). Now consider an arbitrary pair $(x,y) \in X^2$. If $x = y$, T-E-antisymmetry is trivially fulfilled. Without loss of generality, assume $x \prec y$ (otherwise, swap x and y and apply the same arguments), and we obtain

$$T\left(L_n(x,y), L_n(y,x)\right) = T(\alpha, \beta_n) \leq \sup_{n\in\mathbb{N}} T(\alpha, \beta_n) = \gamma = E(x,y).$$

Let us now consider the T-transitivity of the family $(L_n)_{n\in\mathbb{N}}$. For that purpose, consider a triple $(x,y,z) \in X^3$. If any two of these three elements are equal, T-

transitivity is trivial. Suppose, therefore, that x, y, and z are pairwise different. Since $\preceq$ is a linear ordering, it is sufficient to consider the following six cases:

$x \prec y \prec z$: $\quad T\big(L_n(x,y), L_n(y,z)\big) = T(\alpha,\alpha) \leq \alpha = L_n(x,z)$
$x \prec z \prec y$: $\quad T\big(L_n(x,y), L_n(y,z)\big) = T(\alpha,\beta_n) \leq \alpha = L_n(x,z)$
$y \prec x \prec z$: $\quad T\big(L_n(x,y), L_n(y,z)\big) = T(\beta_n,\alpha) \leq \alpha = L_n(x,z)$
$y \prec z \prec x$: $\quad T\big(L_n(x,y), L_n(y,z)\big) = T(\beta_n,\alpha) \leq \beta_n = L_n(x,z)$
$z \prec x \prec y$: $\quad T\big(L_n(x,y), L_n(y,z)\big) = T(\alpha,\beta_n) \leq \beta_n = L_n(x,z$
$z \prec y \prec x$: $\quad T\big(L_n(x,y), L_n(y,z)\big) = T(\beta_n,\beta_n) \leq \beta_n = L_n(x,z)$

We have shown, therefore, that all L_n are T-E-orderings. Since the sequence β_n is linearly ordered, L_n is a linearly ordered sequence of T-E-orderings. It is clear that the smallest possible upper bound of $(L_n)_{n\in\mathbb{N}}$ is given by the following fuzzy relation:

$$\tilde{L}(x,y) = \begin{cases} 1 & \text{if } x = y \\ \alpha & \text{if } x \prec y \\ \sup\limits_{n\in\mathbb{N}} \beta_n & \text{if } x \succ y \end{cases}$$

Taking an arbitrary pair $(x,y) \in X^2$ fulfilling $x \prec y$, we obtain that T-E-antisymmetry is violated (by (5.2)):

$$T\big(\tilde{L}(x,y), \tilde{L}(y,x)\big) = T(\alpha, \sup_{n\in\mathbb{N}} \beta_n) > \sup_{n\in\mathbb{N}} T(\alpha,\beta_n) = \gamma = E(x,y)$$

Any upper bound for the sequence $(L_n)_{n\in\mathbb{N}}$, therefore, violates T-E-antisymmetry. Hence, the sequence $(L_n)_{n\in\mathbb{N}}$ has no upper bound in $\text{ext}(L)$ and, therefore, no supremum. □

Proposition 24 particularly implies that we may run into a situation where Zorn's Lemma is not applicable if we consider a t-norm which is not left-continuous. Since, as we will see later, Zorn's Lemma is most often the key to extension theorems à la Szpilrajn, it is unavoidable to *restrict to left-continuous t-norms for the remaining parts of the paper*. It is worth to mention that this is not a serious restriction in practical terms. Triangular norms that are not left-continuous not even allow to build up a most basic structure of many-valued logics—*GL*-monoids [12, 14, 16, 18, 19, 21]. From a strictly logical point of view, therefore, non-left-continuous t-norms make only little sense in our setting anyway.

Many results in this paper are based on constructing counterexamples for a finite subdomain. Before we turn to the actual study of different variants of linearity axioms, we provide a fundamental lemma that allows us to construct counterexamples on a finite subdomain without losing the validity of the counterexample on a larger domain.

Lemma 25 *Assume that we are given a non-empty set X, a T-equivalence E on X, and a T-E-ordering L on X. Then, for any superset $X' \supset X$, there exist fuzzy relations E' and L' such that the following holds:*

(1) E' is a T-equivalence on X' which extends E to whole X in the sense that $E'|_X = E$, i.e. $E'(x,y) = E(x,y)$ for all $x,y \in X$.
(2) L' is a T-E'-ordering on X' which extends L to whole X in the sense $L'|_X = L$, i.e. $L'(x,y) = L(x,y)$ for all $x,y \in X$.

Moreover, if L is a maximal T-E-ordering on X, there exists a maximal T-E'-ordering L'' on X' for which $L''|_X = L$ holds.

Proof. We define

$$E'(x,y) = \begin{cases} E(x,y) & \text{if } x \in X \text{ and } y \in X \\ 1 & \text{if } x = y \notin X \\ 0 & \text{otherwise} \end{cases}$$

$$L'(x,y) = \begin{cases} L(x,y) & \text{if } x \in X \text{ and } y \in X \\ 1 & \text{if } x = y \notin X \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that E' is indeed a T-equivalence on X' and L' is a T-E'-ordering on X'. The extension properties $E'|_X = E$ and $L'|_X = L$ are also trivial by the above definitions. Now assume that L is a maximal T-E-ordering on X. Now apply the above construction of E' and L'. Therefore, by Theorem 23, there exists a T-E'-ordering L'' on X' which is a maximal extension of L'. Trivially, the restriction of $L''|_X$ must be a T-E-ordering which extends L. Then $L''|_X = L$ has to hold, otherwise L would not be a maximal T-E-ordering on X. □

6 Strong Completeness

A simple concept of completeness of fuzzy relations which is common in fuzzy preference modeling [1, 8, 9, 13] is based on replacing the crisp disjunction in (2.1) by the maximum t-conorm.

Definition 26 A binary fuzzy relation R on X is called *strongly complete* if and only if the following holds for all $x,y \in X$:

$$\max\big(R(x,y), R(y,x)\big) = 1$$

A unique characterization of T-E-orderings fulfilling strong completeness is available, which we repeat first.

Definition 27 Let $\lesssim$ be a crisp ordering on X and let E be a fuzzy equivalence relation on X. E is called *compatible with* $\lesssim$, if and only if the following implication holds for all $x,y,z \in X$:

$$x \lesssim y \lesssim z \Rightarrow E(x,z) \leq \min\big(E(x,y),E(y,z)\big)$$

Compatibility between a crisp ordering $\lesssim$ and a fuzzy equivalence relation E can be interpreted as follows: The two outer elements of a three-element chain are at least as distinguishable as any two inner elements.

Theorem 28 [2,3] *Consider a fuzzy relation L on a domain X and a T-equivalence E. Then the following two statements are equivalent:*

(i) L is a strongly complete T-E-ordering.
(ii) There exists a linear ordering $\lesssim$ the relation E is compatible with such that L can be represented as follows:

$$L(x,y) = \begin{cases} 1 & \text{if } x \lesssim y \\ E(x,y) & \text{otherwise} \end{cases} \tag{6.1}$$

As an important consequence of Theorem 28, we obtain that strong completeness implies maximality.

Proposition 29 *For any T-equivalence E, all strongly complete T-E-orderings are maximal.*

Proof. Consider an arbitrary strongly complete T-E-ordering L. Assume that there is a extension of L denoted L' which is non-trivial, i.e. there is a pair (a,b) such that $L'(a,b) > L(a,b)$. Since $L(a,b)$, by representation (6.1) is either 1 or $E(a,b)$, $L(a,b) = E(a,b) < 1$ must hold, otherwise $L'(a,b) > L(a,b)$ could not be satisfied. Since $\lesssim$ is linear, $b \lesssim a$ must hold, implying $L(b,a) = 1$ and we obtain

$$T\big(L'(a,b),L'(b,a)\big) = T\big(L'(a,b),1\big) = L'(a,b) > L(a,b) = E(a,b)$$

which contradicts T-E-antisymmetry; hence L must be maximal. □

Now the question is whether the reverse implication holds, too. The answer, however, is negative, at least if we consider a t-norm which is smaller than the minimum.

Proposition 30 *Assume that $T \neq T_{\mathbf{M}}$, Then, for any set X with at least two elements, there exists a T-equivalence E and a T-E-ordering L for which no strongly complete extension exists.*

Proof. Let us first consider a two-element set $\{a,b\} \subseteq X$. Since $T \neq T_{\mathbf{M}}$, there exist two values $\alpha, \beta \in (0,1)$ such that

$$T(\alpha,\beta) < \min(\alpha,\beta).$$

Now we construct the following two relations

$$L = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}$$

where we choose $\gamma = T(\alpha,\beta)$. It is easy to prove that E is a T-equivalence on $\{a,b\}$ and that L is a T-E-ordering $\{a,b\}$. If there was a strongly complete extension of L, it would be possible to lift either $L(a,b)$ or $L(b,a)$ to 1. Therefore, the following two relations are the two smallest possible fuzzy relations which extend L and which are strongly complete:

$$L^* = \begin{pmatrix} 1 & 1 \\ \beta & 1 \end{pmatrix} \qquad L^\circ = \begin{pmatrix} 1 & \alpha \\ 1 & 1 \end{pmatrix}$$

However, we obtain that both already violate T-E-antisymmetry:

$$T\big(L^*(a,b), L^*(b,a)\big) = T(1,\beta) = \beta \geq \min(\alpha,\beta) > T(\alpha,\beta) = \gamma = E(a,b)$$
$$T\big(L^\circ(a,b), L^\circ(b,a)\big) = T(\alpha,1) = \alpha \geq \min(\alpha,\beta) > T(\alpha,\beta) = \gamma = E(a,b)$$

Therefore, L cannot have a strongly complete extension. By Lemma 25, we can extend E and L to a T-equivalence E' and a T-E'-ordering L', respectively, which are both binary fuzzy relations on whole X. If L' had a strongly complete extension, this would imply that also L has a strongly complete extension—which has already been proved not to be the case. Therefore, L' is a T-E'-ordering on X which does not have a strongly complete extension. □

Proposition 30 states that [SZP] is not fulfillable for strong completeness if $T \neq T_{\mathbf{M}}$. Trivially, if we have a T-E-ordering L for which no strongly complete extension exists, [INT] cannot hold either, since it is not possible to represent L as the intersection of strongly complete extensions if such extensions do not exist. Moreover, [MAX] does not hold either, since a maximal extension exists for all L (by Proposition 23), even for those for which no strongly complete extension exists.

It remains open so far whether the same problems occur if $T = T_{\mathbf{M}}$ is considered. The following fundamental lemma provides the basis for a full answer.

Lemma 31 *Consider a $T_{\mathbf{M}}$-equivalence E. A T-E-ordering is maximal if and only if it is strongly complete.*

Proof. Let us assume that L is a $T_{\mathbf{M}}$-E-ordering which is maximal but not strongly complete, i.e. there exists a pair (a,b) such that $L(a,b) < 1$ and $L(b,a) < 1$. Without loss of generality suppose that $L(a,b) \leq L(b,a)$, otherwise we can swap a and b. Now we define the following binary fuzzy relation on X:

$$L'(x,y) = \max\big(L(x,y), \min(L(x,b), L(a,y))\big)$$

L' is E-reflexive, since it trivially extends L. In order to prove $T_{\mathbf{M}}$-E-antisymmetry, we have to consider the following inequalities (taking into account that min and max distribute):

$$\begin{aligned} T_{\mathbf{M}}\big(L'(x,y), L'(y,x)\big) &= \min\Big(\max\big(L(x,y), \min(L(x,b), L(a,y))\big), \\ &\qquad \max\big(L(y,x), \min(L(y,b), L(a,x))\big)\Big) \\ &= \max\Big(\min\big(L(x,y), L(y,x)\big), \\ &\qquad \min\big(L(x,y), L(y,b), L(a,x)\big), \\ &\qquad \min\big(L(y,x), L(x,b), L(a,y)\big), \\ &\qquad \min\big(L(x,b), L(a,y), L(y,b), L(a,x)\big)\Big) \end{aligned}$$

It is sufficient to show that no argument of the above maximum exceeds $E(x,y)$, which is trivial for $\min\big(L(x,y), L(y,x)\big)$, since L is $T_{\mathbf{M}}$-E-antisymmetric. In order to show the three other inequalities, let us mention that the equality

$$E(x,y) = \min\big(L(x,y), L(y,x)\big)$$

holds (which is easy to prove by merging E-reflexivity and $T_{\mathbf{M}}$-E-antisymmetry). That implies that either $L(x,y)$ or $L(y,x)$ must equal $E(x,y)$. Let us first assume that $L(x,y) = E(x,y)$. This immediately implies that

$$\min\big(L(x,y), L(y,b), L(a,x)\big) \leq E(x,y).$$

Moreover, as a consequence of $T_{\mathbf{M}}$-transitivity, we obtain

$$\begin{aligned} \min\big(L(y,x), L(x,b), L(a,y)\big) &\leq L(a,b) = E(a,b), \\ \min\big(L(x,b), L(a,y), L(y,b), L(a,x)\big) &\leq L(a,b) = E(a,b). \end{aligned}$$

If $E(a,b) \leq E(x,y)$ holds, we are done. Otherwise, if $E(a,b) > E(x,y)$ holds, we have

$$\begin{aligned} E(x,y) = L(x,y) &\geq \min\big(L(x,b), L(b,a), L(a,y)\big) \\ &\geq \min\big(L(x,b), E(a,b), L(a,y)\big) = (*) \end{aligned}$$

Since $E(a,b) > E(x,y)$,

$$(*) = \min\big(L(x,b), L(a,y)\big),$$

which then implies

$$\begin{aligned} \min\big(L(y,x), L(x,b), L(a,y)\big) &\leq E(x,y), \\ \min\big(L(x,b), L(a,y), L(y,b), L(a,x)\big) &\leq E(x,y). \end{aligned}$$

The converse case $L(y,x) = E(x,y)$ can be proved analogously.

In order to prove that L' is $T_{\mathbf{M}}$-transitive, let us consider distributivity of minimum and maximum again:

$$\begin{aligned}
T_{\mathbf{M}}\big(L'(x,y),L'(y,z)\big) = \min\Big(&\max\big(L(x,y),\min(L(x,b),L(a,y))\big),\\
&\max\big(L(y,z),\min(L(y,b),L(a,z))\big)\Big)\\
= \max\Big(&\min\big(L(x,y),L(y,z)\big),\\
&\min\big(L(x,y),L(y,b),L(a,z)\big),\\
&\min\big(L(y,z),L(x,b),L(a,y)\big),\\
&\min\big(L(x,b),L(a,y),L(y,b),L(a,z)\big)\Big)
\end{aligned}$$

Then $T_{\mathbf{M}}$-transitivity of L' is guaranteed by the following:

$$\begin{aligned}
\min\big(L(x,y),L(y,z)\big) &\leq L(x,z) \leq L'(x,z)\\
\min\big(L(x,y),L(y,b),L(a,z)\big) &\leq \min\big(L(x,b),L(a,z)\big) \leq L'(x,z)\\
\min\big(L(x,b),L(a,y),L(y,z)\big) &\leq \min\big(L(x,b),L(a,z)\big) \leq L'(x,z)\\
\min\big(L(x,b),L(a,y),L(y,b),L(a,z)\big) &\leq \min\big(L(x,b),L(a,b),L(a,z)\big)\\
&\leq \min\big(L(x,b),L(b,a),L(a,z)\big)\\
&\leq L(x,z) \leq L'(x,z)
\end{aligned}$$

Therefore, L' is a $T_{\mathbf{M}}$-E-ordering which is an extension of L. Since

$$\begin{aligned}
L'(b,a) &= \max\big(L(b,a),\min(L(b,b),L(a,a))\big)\\
&= \max\big(L(b,a),1\big)\\
&= 1 > L(b,a),
\end{aligned}$$

L' is a non-trivial extension of L which contradicts the maximality of L.

The reverse implication—that strong completeness implies maximality—has already been proved by Proposition 29. □

Lemma 31 proves [MAX] for strong completeness for the special case $T = T_{\mathbf{M}}$. As a direct consequence, we obtain that [SZP] holds as well.

Theorem 32 (Szpilrajn Theorem for $T_{\mathbf{M}}$-E-orderings) *Suppose that E is a $T_{\mathbf{M}}$-equivalence. Then any $T_{\mathbf{M}}$-E-ordering has a strongly complete extension.*

Proof. For a given $T_{\mathbf{M}}$-equivalence E and a $T_{\mathbf{M}}$-E-ordering L, Theorem 23 guarantees the existence of a maximal extension of L. Lemma 31 then proves that this extension is strongly complete. □

The above Szpilrajn-like theorem makes inherent use of Zorn's Lemma, therefore, the result is purely existential. Note that, in the case that X is a finite set, there is a constructive answer and an efficient algorithm for computing all possible strongly complete linearizations of a given $T_{\mathbf{M}}$-E-ordering [28].

The question remains whether [INT] can be fulfilled for the case $T = T_{\mathbf{M}}$. The following theorem gives a unique characterization of those $T_{\mathbf{M}}$-E-orderings which can be represented as intersections of strongly complete extensions.

Theorem 33 *Let E be a $T_{\mathbf{M}}$-equivalence on some domain X and let L be a $T_{\mathbf{M}}$-E-ordering. Then the following two statements are equivalent:*

(i) There exists a family of strongly complete $T_{\mathbf{M}}$-E-orderings $(L_i)_{i\in I}$ such that the following representation holds:

$$L(x,y) = \inf_{i\in I} L_i(x,y)$$

(ii) For all $x,y \in X$, $L(x,y) \in \{E(x,y), 1\}$ holds.

Proof.

(i)⇒(ii): Assume that (i) holds, i.e. that L is an intersection of strongly complete extensions. Now choose an arbitrary pair $x,y \in X$. If $L(x,y) = 1$, we are done. Conversely, assume that $L(x,y) < 1$. Then there exists an $i \in I$ such that $L(x,y) \leq L_i(x,y) < 1$ is satisfied. Since L_i is strongly complete, $L_i(y,x) = 1$ must hold. Then, from E-reflexivity and $T_{\mathbf{M}}$-E-antisymmetry of L_i, we obtain

$$E(x,y) \leq L(x,y) \leq L_i(x,y) = \min\big(L_i(x,y), L_i(y,x)\big) \leq E(x,y),$$

i.e. that $L(x,y) = E(x,y)$ holds.

(ii)⇒(i): We choose the index set in the following way:

$$I = \{(a,b) \in X^2 \mid L(a,b) = E(a,b)\}$$

Then, for any pair $(a,b) \in I$,

$$E(a,b) = L(a,b) \leq L(b,a)$$

holds. Now we define a fuzzy relation $L_{a,b}$ as

$$L_{a,b}(x,y) = \max\big(L(x,y), \min(L(x,b), L(a,y))\big).$$

Analogously to the proof of Lemma 31, $L_{a,b}$ is a $T_{\mathbf{M}}$-E-ordering which is an extension of L and the following holds:

$$L_{a,b}(a,b) = \max\big(L(a,b), \min(L(a,b), L(a,b))\big) = L(a,b) = E(a,b) \qquad (6.2)$$

$$L_{a,b}(b,a) = \max\big(L(b,a), \min(L(b,b), L(a,a))\big) = 1 \qquad (6.3)$$

By Theorem 32, there exists at least one strongly complete extension of $L_{a,b}$. Let us choose one such extension and denote it with $L^*_{a,b}$. Since $L^*_{a,b}$ fulfills $T_{\mathbf{M}}$-E-antisymmetry as well,

$$E(a,b) \geq \min\left(L^*_{a,b}(a,b), L^*_{a,b}(b,a)\right) = \min\left(L^*_{a,b}(a,b), 1\right) = L^*_{a,b}(a,b) \geq E(a,b)$$

must hold. The proof is completed if we succeed to prove the equality

$$L(x,y) = \inf_{(a,b)\in I} L^*_{a,b}(x,y).$$

Let us choose an arbitrary pair (x,y). If $L(x,y) = 1$, then $L^*_{a,b}(x,y) = 1$ must hold as well, since all $L^*_{a,b}$ are extension of L. Conversely, if $L(x,y) = E(x,y)$ holds, then we have $(x,y) \in I$ and, by (6.2),

$$E(x,y) = L(x,y) = L^*_{x,y}(x,y) \geq \inf_{(a,b)\in I} L^*_{a,b}(x,y).$$

The reverse inequality

$$L(x,y) \leq \inf_{(a,b)\in I} L^*_{a,b}(x,y)$$

holds in any case, since all $L^*_{a,b}$ are extensions of L. □

In order to complete the answer to the question whether [INT] holds for strong completeness in the case $T = T_{\mathbf{M}}$, let us consider a domain X with at least two elements and choose two different elements a and b from X. Moreover, we choose two values $\alpha, \beta \in (0,1)$ such that $\alpha < \beta$ holds and define the following two binary fuzzy relations:

$$L(x,y) = \begin{cases} 1 & \text{if } x = y \\ \alpha & \text{if } (x,y) = (a,b) \\ \beta & \text{if } (x,y) = (b,a) \\ 0 & \text{otherwise} \end{cases} \qquad E(x,y) = \begin{cases} 1 & \text{if } x = y \\ \alpha & \text{if } \{x,y\} = \{a,b\} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to prove that E is a $T_{\mathbf{M}}$-equivalence and that L is a $T_{\mathbf{M}}$-E-ordering. Obviously, $L(b,a) = \beta \notin \{E(b,a), 1\} = \{\alpha, 1\}$, therefore, Theorem 33 implies that L cannot be represented as an intersection of strongly complete intersections, which finally proves that [INT] does not hold for strong completeness in the case $T = T_{\mathbf{M}}$ either.

Remark 34 Note that [INT] holds in a weak sense for all left-continuous t-norms T. In [33], it is proved that the following two statements are equivalent for all binary fuzzy relations L on a domain X:

(i) L is a T-preordering.
(ii) There exists a family of $X \to [0,1]$ mappings $(f_i)_{i\in I}$ such that the following representation holds:

$$L(x,y) = \inf_{i\in I} \vec{T}\left(f_i(x), f_i(y)\right) \tag{6.4}$$

It is easy to see (by Lemma 17) that all components of the intersection

$$\vec{T}\left(f_i(x), f_i(y)\right)$$

are strongly complete T-preorderings (a type of binary fuzzy relations often called *fuzzy weak orderings* [1,8,13]). Since a T-E-ordering is obviously a T-preordering, this representation holds in our framework, too. However, the results in this section have shown, that it is not possible in general to represent a T-E-ordering as the intersection of strongly complete fuzzy relations which also fulfill all three axioms of T-E-orderings.

7 T-Linearity

In this subsection, we consider a type of fuzzy completeness which is based on the idea of generalizing (2.2) to the fuzzy case by replacing the Boolean complement by the negation N_T induced by the residual implication of the underlying left-continuous t-norm T [19].

Definition 35 Let T be a left-continuous t-norm. A binary fuzzy relation is called *T-linear* if and only if the following holds for all $x,y \in X$:

$$N_T\left(L(x,y)\right) = \vec{T}\left(L(x,y),0\right) \leq L(y,x)$$

The following fundamental theorem provides the basis for proving that [SZP] and [INT] are preserved for T-linearity.

Theorem 36 [19] *Consider a T-equivalence E, and a T-E-ordering L. Then, for any pair $(a,b) \in X^2$, there exists a T-linear extension $L_{a,b} \supseteq L$ which fulfills $L(a,b) = L_{a,b}(a,b)$.*

As a trivial consequence we obtain an appropriate linearization theorem, i.e. a result showing that [SZP] holds for T-linearity.

Corollary 37 (Szpilrajn Theorem for T-linearity) [19] *Given a T-equivalence E, any T-E-ordering has a T-linear extension.*

Moreover, as another consequence of Theorem 36, we can also show that [INT] holds for T-linearity, too.

Corollary 38 [19] *Consider a T-equivalence E. Then, for any T-E-ordering L, there exists a family of T-linear T-E-orderings $(L_i)_{i\in I}$ such that L can be represented as the intersection of all L_i, i.e., for all $x,y \in X$,*

$$L(x,y) = \inf_{i\in I} L_i(x,y).$$

Proof. We choose $I = X^2$, and for every $(a,b) \in I$, let $L_{a,b}$ denote an extension of L for which $L(a,b) = L_{a,b}(a,b)$ holds (existence guaranteed by Theorem 36). Then the equality

$$L(x,y) \leq \inf_{(a,b)\in I} L_{a,b}(x,y) \leq L_{x,y}(x,y) = L(x,y)$$

must hold, which completes the proof. □

It remains to clarify the correspondence between T-linearity and maximality.

Corollary 39 *Let E be a T-equivalence and L be a T-E-ordering L. If L is maximal, i.e.* $\mathrm{ext}(L) = \{L\}$, *then L is T-linear.*

Proof. Assume that L is maximal. By Corollary 37, there exists a T-linear extension of L. Since L is its only possible trivial extension, L must be T-linear. □

As we will see next, however, the reverse does not hold in general which implies that the fundamental property [MAX] cannot be preserved for T-linearity.

Proposition 40 *For all domains X with at least two elements, there exists a T-equivalence E and a T-E-ordering L which fulfills T-linearity, but which is not maximal.*

Proof. Let us choose an arbitrary $\alpha \in (0,1)$. Due to (2) in Lemma 17, $N_T(\alpha) < 1$ holds. If $N_T(\alpha) \leq \alpha$ holds, denote $\beta = \alpha$, otherwise choose $\beta = N_T(\alpha)$. In the latter case,

$$\beta = N_T(\alpha) > \alpha$$

which implies ($\vec{T}$ is non-increasing in its first argument; cf. Lemma 17)

$$N_T(\beta) \leq N_T(\alpha) = \beta.$$

Hence, in any case, $N_T(\beta) \leq \beta$ holds. Now let us define the following two fuzzy relations:

$$L(x,y) = \begin{cases} 1 & \text{if } x = y \\ \beta & \text{otherwise} \end{cases} \qquad E(x,y) = L(x,y)$$

It is trivial to prove that E is a T-equivalence and that L is a T-E-ordering. Since $N_T(\beta) \leq \beta$ holds, L is T-linear. Since we can choose an arbitrary linear ordering $\preceq$ on X (existence again guaranteed by the classical Szpilrajn Theorem), we can define the following fuzzy relation;

$$L'(x,y) = \begin{cases} 1 & \text{if } x \preceq y, \\ \beta & \text{otherwise.} \end{cases}$$

It is routine matter to prove that L' is a T-E-ordering which is obviously a non-trivial extension of L. Therefore, L cannot be maximal. □

Nonchalantly speaking, this means that T-linearity is, in any case, a property that is "strictly weaker" than maximality. This is particularly true if the t-norm T does not have zero divisors. Let us consider the negation N_T:

$$N_T(x) = \vec{T}(x,0) = \sup\{u \in [0,1] \mid T(u,x) \leq 0\}$$

If $x > 0$, then $u = 0$ is the only value for which $T(u,x) = 0$ can hold. Therefore, any t-norm without zero divisors induces the so-called *intuitionistic negation*, also known as *Gödel negation*:

$$N_{\mathbf{I}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

In such a case, T-linearity only means that, for a fixed pair $(x,y) \in X^2$, $L(x,y) = 0$ implies $L(y,x) = 1$; however, if $\min\big(L(x,y), L(y,x)\big) > 0$, $L(x,y)$ and $L(y,x)$ may take any values from $(0,1]$ without violating T-linearity.

8 S-Completeness

Now we study a generalization of strong completeness which is also well-known in fuzzy preference modeling [13]. It is simply based on replacing the disjunction in (2.1) by a general t-conorm.

Definition 41 Let S be a t-conorm. A binary fuzzy relation R on X is called *S-complete* if and only if the following holds for all $x,y \in X$:

$$S\big(R(x,y), R(y,x)\big) = 1$$

In principle, it is possible to consider any t-conorm S. Since we are examining the completeness axioms in the framework of fuzzy orderings, it seems reasonable (and this is also usual even in more general settings in fuzzy preference modeling) to assume a certain structural compatibility between the underlying t-norm T and the t-conorm under consideration. For the remaining section, therefore, assume that (T,S,N) is a de Morgan triple for some strong negation N.

As the first important result, we obtain a full answer to all our questions for the case that T does not have zero divisors.

Lemma 42 *Provided that T does not have zero divisors, S-completeness is equivalent to strong completeness.*

Proof. Assume that an arbitrary binary fuzzy relation R is S-complete, i.e.

$$S\big(R(x,y),R(y,x)\big) = N\Big(T\left(N(R(x,y)),N(R(y,x))\right)\Big) = 1.$$

Since N is a strictly decreasing continuous bijection,

$$T\left(N(R(x,y)),N(R(y,x))\right) = 0$$

holds. Since T does not have zero divisors, either $N(R(x,y)) = 0$ or $N(R(y,x)) = 0$ must hold, which implies either $R(x,y) = 1$ or $R(y,x) = 1$. The fuzzy relation R, therefore, is strongly complete.

The reverse implication follows trivially from the fact that the maximum is the smallest possible t-conorm, therefore, strong completeness is a stronger property than S-completeness. □

Proposition 43 *Assume that T does not have zero divisors. In the case $T = T_{\mathbf{M}}$, the properties [SZP] and [MAX] hold. If $T \neq T_{\mathbf{M}}$, none of the three fundamental properties holds.*

Proof. Trivial by Lemma 42 and the results in Section 6. □

In particular, this entails that S-completeness does not allow any of the three fundamental properties for strict t-norms—including the important product $T_{\mathbf{P}}$. Now let us approach t-norms with zero divisors. In the first step, we consider t-norms inducing a strong negation.

Lemma 44 *Consider a t-norm T such that N_T is a strong negation. Provided that S is chosen as*

$$S(x,y) = N_T\left(T(N_T(x),N_T(y))\right), \tag{8.1}$$

then S-completeness is equivalent to T-linearity.

Proof. Consider a binary fuzzy relation R and fix two arbitrary elements $x,y \in X$. Then, for the pair (x,y), (8.1) is equivalent to

$$N_T\Big(T\big(N_T(R(x,y)),N_T(R(y,x))\big)\Big) = 1.$$

Since N_T is a strong negation, this is equivalent to

$$T\big(N_T(R(x,y)),N_T(R(y,x))\big) = 0. \tag{8.2}$$

Due to the residuation principle, i.e. (1) in Lemma 17, (8.2) is equivalent to

$$N_T(R(x,y)) \leq \vec{T}\big(N_T(R(y,x)),0\big) = N_T\big(N_T(R(y,x))\big). \tag{8.3}$$

Since N_T is supposed to be strong, we finally obtain that (8.3) is equivalent to

$$N_T(R(x,y)) \leq N_T\big(N_T(R(y,x))\big) = R(y,x)$$

which is exactly T-linearity for the pair (x,y). Since this holds for all pairs (x,y), S-completeness is equivalent to T-linearity. □

Theorem 45 *Under the assumption that N_T is a strong negation and that we use $N = N_T$, the fundamental properties [SZP] and [INT] hold for S-completeness.*

Proof. Follows directly from Lemma 44 and the results in Section 7. □

The class of t-norms inducing a strong negation includes all nilpotent t-norms (cf. Lemma 20), most importantly the Łukasiewicz t-norm $T_{\mathbf{L}}$. Moreover, Theorem 45 is also applicable to so-called *nilpotent Zadeh triples* [6, 7], i.e. de Morgan triples (T_N, S_N, N) where N is a strong negation and where T_N and S_N are defined as follows:

$$T_N(x,y) = \begin{cases} \min(x,y) & \text{if } y > N(x) \\ 0 & \text{otherwise} \end{cases}$$
$$S_N(x,y) = \begin{cases} \max(x,y) & \text{if } x < N(y) \\ 1 & \text{otherwise} \end{cases}$$

This class also comprises the nilpotent minimum $T_{\mathbf{nM}}$ for $N(x) = 1 - x$.

It remains to study what happens if T does have zero divisors and if $N \neq N_T$ ($N = N_T$ can only be fulfilled if T induces a strong negation, anyway). The following theorem provides a sufficient condition for the fulfillment of [SZP] and [INT].

Theorem 46 *Consider a T-equivalence E. If $N \leq N_T$ holds, i.e. $N(x) \leq N_T(x)$ for all $x \in [0,1]$, S-completeness fulfills [SZP] and [INT].*

Proof. Consider an arbitrary T-linear T-E-ordering L on X, i.e., for all $x,y \in X$,

$$N_T\big(L(x,y)\big) \leq L(y,x).$$

Assuming that $N \leq N_T$ holds, we obtain (using (5) from Lemma 17)

$$\begin{aligned} T\big(N(L(x,y)),N(L(y,x))\big) &\leq T\big(N_T(L(x,y)),N_T(L(y,x))\big) \\ &\leq T\big(L(y,x),\vec{T}(L(y,x),0)\big) = 0 \end{aligned}$$

which implies

$$S\big(L(x,y),L(y,x)\big) = N\Big(T\big(N(L(x,y)),N(L(y,x))\big)\Big) = N(0) = 1.$$

Therefore, $N \leq N_T$ is a sufficient condition that T-linearity implies S-completeness. Then [SZP] is satisfied, as Corollary 37 guarantees that any T-E-ordering has a T-linear extension which is automatically S-complete if $N \leq N_T$ holds. Analogously, any T-E-ordering can be represented as the intersection of T-linear T-E-orderings (cf. Corollary 38). If $N \leq N_T$, the components of this intersection are also S-complete. Hence, also [INT] is satisfied in case that $N \leq N_T$. □

It remains to clarify whether $N \leq N_T$ is also a necessary condition for the fulfillment of [SZP] and [INT] by S-completeness.

Lemma 47 *Assume that X has at least two elements. If there is an $\alpha \in (0,1)$ such that $N(\alpha) > N_T(\alpha)$ and additionally $N_T(N_T(\alpha)) = \alpha$ holds, there exists a T-equivalence E and a maximal T-E-ordering L which is not S-complete.*

Proof. $N(\alpha) > N_T(\alpha)$ implies

$$\alpha = N(N(\alpha)) < N(N_T(\alpha)).$$

Now consider an arbitrary two-element set $\{a,b\} \subseteq X$ and define the following two fuzzy relations on $\{a,b\}$:

$$L = \begin{pmatrix} 1 & N_T(\alpha) \\ \alpha & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Obviously, E is the crisp equality which is trivially a T-equivalence. It is, moreover, easy to see that L is a T-E-ordering, where T-E-antisymmetry follows from (5) in Lemma 17:

$$T\big(L(a,b),L(b,a)\big) = T\big(\alpha,N_T(\alpha)\big) = T\big(\alpha,\vec{T}(\alpha,0)\big) = 0$$

Now consider the following:

$$T\big(N(L(a,b)),N(L(b,a))\big) = T\big(N(\alpha),N(N_T(\alpha))\big) \geq T\big(N(\alpha),\alpha\big) = (*)$$

Since $N_T(\alpha) = \vec{T}(\alpha, 0)$ is the largest value β for which $T(\alpha, \beta) = 0$ holds and $N(\alpha) > N_T(\alpha)$, we obtain that $(*) > 0$; therefore,

$$S\big(L(a,b), L(b,a)\big) = N\Big(T\big(N(L(a,b)), N(L(b,a))\big)\Big) < N(0) = 1,$$

i.e. L is a T-E-ordering which is not S-complete. Once again recall that $N_T(\alpha) = \vec{T}(\alpha, 0)$ is the largest value β for which $T(\alpha, \beta) = 0$. That implies that no larger value for $L(a,b)$ than $N_T(\alpha)$ can be chosen without violating T-E-antisymmetry. Analogously, $\alpha = N_T(N_T(\alpha)) = \vec{T}(N_T(\alpha), 0)$ is absolutely the largest value β for which $T(\beta, N_T(\alpha)) = 0$. Hence, no larger value for $L(b,a)$ than α may be chosen without harming T-E-antisymmetry. Therefore, L is a maximal T-E-ordering on $\{a,b\}$. By Lemma 25, we are able to extend L to a maximal T-E'-ordering L' for which $L'|_{\{a,b\}} = L$ holds, where E' denotes the crisp equality on X. Then L' cannot be S-complete either, since S-completeness is already violated for the pair (a,b). □

Theorem 48 *Let X have at least two elements. If there is an $\alpha \in (0,1)$ such that $N(\alpha) > N_T(\alpha)$ and additionally $N_T(N_T(\alpha)) = \alpha$ holds, S-completeness fulfills none of the three fundamental properties.*

Proof. With the assumptions that there is an $\alpha \in (0,1)$ such that $N(\alpha) > N_T(\alpha)$ and that $N_T(N_T(\alpha)) = \alpha$ is satisfied, Lemma 47 states that there is a T-equivalence E and a maximal T-E-ordering L which is not S-complete, which already disproves [MAX]. What particularly follows is that no S-complete extension exists for this L which disproves [SZP]. The fundamental property [INT] cannot be satisfied either if there exists no S-complete extension at all. □

The additional requirement $N_T(N_T(\alpha)) = \alpha$ in Lemma 47 and Theorem 48 is not as strong as it might appear at first glance. First of all, if the underlying t-norm T induces a strong negation, this requirement is fulfilled anyway. This also implies that [SZP] and [INT] are fulfilled for Łukasiewicz triples if and only if $N \leq N_T$. The same is true for nilpotent Zadeh triples.

This line of argumentation is even valid for all continuous t-norms with zero divisors that are not nilpotent. It is easy to see taking Theorem 10 into account that a continuous t-norm can only have zero divisors if there is a summand $\langle a_i, e_i, T_i \rangle$ such that $a_i = 0$ and T_i is nilpotent. If T itself is not nilpotent, $e_i < 1$ must hold and N_T obeys the following representation:

$$N_T(x) = \begin{cases} 1 & \text{if } x = 0 \\ e_i \cdot N_{T_i}\left(\frac{x}{e_i}\right) & \text{if } x \in (0, e_i) \\ 0 & \text{otherwise} \end{cases}$$

For any strong negation N, which is of course a strictly decreasing continuous mapping, there exists an $\alpha \in (0, e_i)$ such that $N(\alpha) > e_i$. As easy to see,

$$N_T(\alpha) = e_i \cdot N_{T_i}(\frac{\alpha}{e_i}) \leq e_i < N(\alpha).$$

Taking into account that N_{T_i} is a strong negation (since T_i is nilpotent), we obtain

$$N_T\big(N_T(\alpha)\big) = e_i \cdot N_{T_i}\left(\frac{e_i \cdot N_{T_i}\big(\frac{\alpha}{e_i}\big)}{e_i}\right) = e_i \cdot N_{T_i}\Big(N_{T_i}\Big(\frac{\alpha}{e_i}\Big)\Big) = e_i \cdot \frac{\alpha}{e_i} = \alpha.$$

Hence, if T is a continuous t-norm with zero divisors that is not nilpotent, the conditions of Lemma 47 and Theorem 48, respectively, can be satisfied which implies that none of the three fundamental properties can hold.

If T, no matter whether continuous or not, does not have zero divisors, the inequality $N_T(N_T(\alpha)) = \alpha$ can never be fulfilled for an $\alpha \in (0,1)$; instead, Proposition 43 clarifies the whole situation in an exhaustive way.

We can summarize these findings about continuous t-norms in the following way:

- If $T = T_{\mathbf{M}}$, S-completeness is equivalent to strong completeness and the fundamental properties [SZP] and [MAX] are fulfilled.
- If T is nilpotent, S-completeness fulfills [SZP] and [INT] if and only if $N \leq N_T$.
- For all other continuous t-norms, none of the three fundamental properties can be fulfilled.
- Some questions remain open for non-continuous, but left-continuous, t-norms with zero divisors that do not induce a strong negation. Such t-norms exist of course [22], but they can be considered rather exotic objects of minor practical relevance.

9 Maximality

We are now in the following situation: strong completeness implies maximality, but not vice versa (except for $T = T_{\mathbf{M}}$; cf. Propositions 29 and 30 and Lemma 31); maximality implies T-linearity, but not vice versa (cf. Corollary 39 and Proposition 40). On the one hand, this entails that strong completeness is too strong a property to fulfill any fundamental properties (except for $T = T_{\mathbf{M}}$). On the other hand, T-linearity fulfills [SZP] and [INT], but is too weak a property to fulfill [MAX]. It remains unclear whether there is an appropriate concept of fuzzy linearity/completeness "between" strong completeness and T-linearity which maintains all three fundamental properties. As fulfillment of [MAX] would be nothing else but the equivalence of maximality with this respective property, we can now treat maximality as a concept of fuzzy linearity/completeness in its own right. It is clear then by Theorem 23 that

[SZP] is guaranteed to be fulfilled. The only problem remains whether maximality has a reasonable axiomatization, i.e. a simple criterion which allows to check whether a given T-E-ordering is maximal or not.

It is clear that a natural upper bound for extensions of a T-E-ordering L is given by T-E-antisymmetry. Following this line of thought, we are able to provide a sufficient condition for maximality.

Lemma 49 *Let E be a T-equivalence. If a T-E-ordering L fulfills*

$$L(x,y) = \vec{T}\left(L(y,x), E(x,y)\right) \tag{9.1}$$

for some pair $(x,y) \in X^2$, every extension $L' \supseteq L$ fulfills $L'(x,y) = L(x,y)$.

Proof. T-E-antisymmetry implies by the residuation principle (Equivalence (1) in Lemma 17) that

$$L(x,y) \leq \vec{T}\left(L(y,x), E(x,y)\right).$$

Using the definition of $\vec{T}$, Equality (9.1) can be written as

$$L(x,y) = \sup\{u \in [0,1] \mid T(u, L(y,x)) \leq E(x,y)\}.$$

Assume a non-trivial extension $L' \supseteq L$ to exist. If $L'(x,y) > L(x,y)$ held, this would imply that

$$L'(x,y) > \sup\{u \in [0,1] \mid T(u, L(y,x)) \leq E(x,y)\},$$

i.e. that

$$T\left(L'(x,y), L'(y,x)\right) \geq T\left(L'(x,y), L(y,x)\right) > E(x,y).$$

This contradicts to T-E-antisymmetry. Hence, $L'(x,y) = L(x,y)$ must hold. □

Theorem 50 *If a T-E-ordering L fulfills* (9.1) *for all pairs $(x,y) \in X^2$, L is guaranteed to be maximal.*

Proof. Follows trivially from Lemma 49 by considering all pairs (x,y). □

The question remains whether the reverse is true as well, i.e. whether the fulfillment of condition (9.1) for all $x, y \in X$ is also a necessary condition for maximality. The following theorem gives a negative answer. Even worse, we obtain that maximality cannot be axiomatized in the usual way by considering pairs of elements only.

Theorem 51 *Consider a domain X with at least four elements and assume that there exists a value $\alpha \in (0,1)$ such that*

$$\alpha = \vec{T}\left(\alpha, T(\alpha,\alpha)\right). \tag{9.2}$$

Then there exists a T-equivalence E such that maximality of T-E-orderings is not decidable by considering pairs of elements only.

Proof. Let us denote $\beta = T(\alpha,\alpha)$. We consider an arbitrary four-element subset $X' = \{a,b,c,d\} \subseteq X$ and define the following two binary fuzzy relations on X':

$$L = \begin{pmatrix} 1 & 1 & \alpha & \alpha \\ \beta & 1 & \alpha & \alpha \\ \alpha & \alpha & 1 & \beta \\ \alpha & \alpha & 1 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & \beta & \beta & \beta \\ \beta & 1 & \alpha & \beta \\ \beta & \alpha & 1 & \beta \\ \beta & \beta & \beta & 1 \end{pmatrix}$$

It is easy, yet tedious, to check that E is a T-equivalence and that L is a T-E-ordering on X'. Now we prove that L is maximal. Suppose that there is a T-E-ordering L' which is a non-trivial extension of L. There are 10 pairs of elements $(x,y) \in \{a,b,c,d\}^2$ for which $L(x,y) < 1$ holds, at least for one of them $L'(x,y) > L(x,y)$ must be satisfied. For the six pairs

$$L(a,c) = L(a,d) = L(b,d) = L(c,a) = L(d,a) = L(d,b) = \alpha$$

we have, taking (9.2) into account,

$$L(x',y') = \alpha = \vec{T}\big(\alpha, T(\alpha,\alpha)\big) = \vec{T}\left(\alpha,\beta\right) = \vec{T}\left(L(y',x'), E(x',y')\right)$$

with $(x',y') \in \{(a,c),(a,d),(b,d),(c,a),(d,a),(d,b)\}$. Then Lemma 49 yields that these six values are maximal and cannot be lifted, since a larger value would immediately deteriorate T-E'-antisymmetry. For $L(b,a) = L(c,d) = \beta$, the same is true, because of $L(a,b) = L(d,c) = 1$. So, only $L'(b,c)$ or $L'(c,b)$ remain to be potentially extensible. Assume that $L'(b,c) = \gamma > \alpha$. L' must be T-transitive, therefore,

$$L'(a,c) \geq T\big(L'(a,b), L'(b,c)\big) = T(1,\gamma) = \gamma > \alpha.$$

This implies

$$T\big(L'(a,c), L'(c,a)\big) \geq T\big(L'(a,c), L(c,a)\big) = T(\gamma,\alpha) = (*)$$

Equality (9.2) can be rewritten as

$$\alpha = \sup\{u \in [0,1] \mid T(u,\alpha) \leq T(\alpha,\alpha) = \beta\}$$

As $\gamma > \alpha$, $(*) > \beta$ must follow, which contradicts T-E-antisymmetry. Analogously, assume $L'(c,b) = \gamma > \alpha$ and consider

$$L'(d,b) \geq T\big(L'(d,c), L'(c,b)\big) = T(1,\gamma) = \gamma > \alpha.$$

Using the same argument as above, it is obtained that $L(c,b)$ cannot be lifted either. Therefore, L is a maximal T-E-ordering on $\{a,b,c,d\}$.

If maximality was decidable by considering pairs of elements only, any restriction of L to a two-element subset of X' would have to be maximal as well. However, this is not the case here, since restricting to $X'' = \{b,c\}$ yields the following:

$$L|_{\{b,c\}} = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} \qquad E|_{\{b,c\}} = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$$

Obviously, $L|_{\{b,c\}}$ is not maximal, since we may lift, e.g., $L|_{\{b,c\}}(b,c)$ to 1 without violating any requirement.

Hence, we have shown that, for any four-element set, there exists a T-equivalence E such that maximality cannot be characterized by an axiom involving only pairs of elements. Now we consider the whole domain X. Lemma 25 guarantees that there is a T-equivalence E'' on X such that $E''|_{\{a,b,c,d\}} = E$ and that there is a maximal T-E''-ordering L'' on X that extends L such that $L''|_{\{a,b,c,d\}} = L$. The restriction of L'' to the subset $\{b,c\}$ gives the same result as shown above, which proves that the maximality of L'' cannot be decided on the basis of considering pairs of elements independently. □

The condition that a value $\alpha \in (0,1)$ fulfilling (9.2) exists is a merely technical prerequisite for the construction of the counterexample in the proof of Theorem 51. First of all, Theorem 51 cannot applicable to $T = T_{\mathbf{M}}$ as strong completeness is an axiomatization of maximality (cf. Lemma 31) that only takes pairs of elements into account. Clearly, the conditions of Theorem 51 cannot be satisfied for $T = T_{\mathbf{M}}$, since, for every value $\alpha \in (0,1)$, $T_{\mathbf{M}}(\alpha,\alpha) = \alpha$ holds, which implies

$$\vec{T}_{\mathbf{M}}\big(\alpha, T_{\mathbf{M}}(\alpha,\alpha)\big) = \vec{T}_{\mathbf{M}}(\alpha,\alpha) = 1.$$

It remains to clarify which other t-norms satisfy this condition. For that purpose, let us consider the following lemma.

Lemma 52 *For a given left-continuous t-norm T, the following two statements are equivalent:*

(1) There exists an $\alpha \in (0,1)$ fulfilling (9.2).
(2) There exists a $\beta \in (0,1)$ such that the mapping $f_\beta(x) = \vec{T}(x,\beta)$ (for $x \in [0,1]$) has a fixed point.

Proof. Let us first assume that an $\alpha \in (0,1)$ fulfilling (9.2) exists. With the setting $\beta = T(\alpha,\alpha)$, $f_\beta(x) = \vec{T}\big(x, T(\alpha,\alpha)\big)$ holds. Then (9.2) implies that α is a fixed point of f_β.

Conversely, assume that a $\beta \in (0,1)$ exists such that mapping f_β has a fixed point. We denote this fixed point with α and $\alpha = f_\beta(\alpha) = \vec{T}(\alpha,\beta)$ holds. Then, on the one hand, (5) in Lemma 17 implies

$$T(\alpha,\alpha) = T\big(\alpha, \vec{T}(\alpha,\beta)\big) \leq \beta. \tag{9.3}$$

On the other hand, applying the residuation principle (cf. (1) in Lemma 17) to the trivial inequality $T(\alpha,\alpha) \leq T(\alpha,\alpha)$ yields that

$$\alpha \leq \vec{T}\big(\alpha, T(\alpha,\alpha)\big),$$

always holds. Now taking into account that the residual implication is non-decreasing in the second argument (see Lemma 17), this implies together with (9.3)

$$\vec{T}\big(\alpha, T(\alpha,\alpha)\big) \leq \vec{T}(x,\beta) = \alpha \leq \vec{T}\big(\alpha, T(\alpha,\alpha)\big)$$

which proves (9.2) and we are done. □

Lemma 52 allows to prove the condition of Theorem 51 for virtually any common t-norm $T \neq T_{\mathbf{M}}$:

- Theorem 51 works for any left-continuous t-norm with zero divisors whose negation N_T has a fixed point (with $\beta = 0$). This particularly includes all t-norms inducing a strong negation, comprising all nilpotent t-norms and nilpotent Zadeh triples.
- If a t-norm T is strict, the representation (4.3) holds, and for any choice of $\beta \in (0,1)$
$$f_\beta(x) = \vec{T}(x,\beta) = \varphi^{-1}\big(\max(\varphi(\beta) - \varphi(x), 0)\big)$$
is a non-increasing continuous mapping with $f_\beta(\beta) = 1$ and $f_\beta(1) = \beta$. Hence, there exists an $\alpha \in (\beta,1)$ which is a fixed point of f_β.
- Assume that T is a ordinal sum $(\langle a_i, e_i, T_i\rangle)_{i\in I}$. If there is at least one left-continuous summand $\langle a_i, e_i, T_i\rangle$ for which an α_i fulfilling (9.2) exists, then choose $\alpha = a_i + (e_i - a_i)\cdot\alpha_i$ and we obtain
$$\begin{aligned}&\vec{T}\big(\alpha, T(\alpha,\alpha)\big)\\ &\quad = \sup\{u \in [0,1] \mid T(u,\alpha) \leq T(\alpha,\alpha)\}\\ &\quad = \sup\{u \in [0,1] \mid T(u, a_i + (e_i - a_i)\cdot\alpha_i) \leq a_i + (e_i - a_i)\cdot T_i(\alpha_i,\alpha_i)\}\\ &\quad = (*)\end{aligned}$$
As $a_i \leq a_i + (e_i - a_i)\cdot T_i(\alpha_i,\alpha_i) \leq e_i$ it is sufficient to consider only values $u \in [a_i, e_i]$, i.e
$$\begin{aligned}(*) &= \sup\{u \in [a_i,e_i] \mid T(u, a_i + (e_i - a_i)\cdot\alpha_i) \leq a_i + (e_i - a_i)\cdot T_i(\alpha_i,\alpha_i)\}\\ &= \sup\{u \in [a_i,e_i] \mid a_i + (e_i - a_i)\cdot T_i\big(\frac{u-a_i}{e_i-a_i},\alpha_i\big) \leq a_i + (e_i - a_i)\cdot T_i(\alpha_i,\alpha_i)\}\\ &= a_i + (e_i - a_i)\cdot\sup\{v \in [0,1] \mid T_i(v,\alpha_i) \leq T_i(\alpha_i,\alpha_i)\}\\ &= a_i + (e_i - a_i)\cdot\vec{T_i}\big(\alpha_i, T_i(\alpha_i,\alpha_i)\big)\\ &= a_i + (e_i - a_i)\cdot\alpha_i,\end{aligned}$$

which proves (9.2) for α. This particularly entails that all continuous t-norms $T \neq T_{\mathbf{M}}$ satisfy the conditions of Theorem 51.

Theorem 51 states that the fundamental property [MAX] cannot be maintained if we consider completeness axioms like (2.1) of (2.2) which both consider pairs of elements only, except for strong completeness in case $T = T_{\mathbf{M}}$. Maximality is a kind of "global" property. In the crisp case, fortunately, maximality remains characterizable by a "local" axiom which only involves pairs of elements. Theorem 51 shows that this nice characterization is lost in the fuzzy case except for the minimum t-norm.

10 Summary and Conclusion

This paper has been concerned with evaluating three concepts of fuzzy linearity/completeness with respect to the three fundamental properties [SZP], [INT], and [MAX]. The findings can be summarized as follows (see Table 1 for a tabular overview):

Strong completeness: this variant provides reasonable results for the minimum t-norm $T_{\mathbf{M}}$. In this case, [SZP] and [MAX] are fulfilled. A characterization of those $T_{\mathbf{M}}$-E-orderings which admit a representation as intersection of strongly complete $T_{\mathbf{M}}$-E-orderings in the sense of the [INT] property has been given. If $T \neq T_{\mathbf{M}}$, none of the fundamental properties is preserved. Strong completeness, therefore, can only serve as an appropriate fuzzy concept of linearity/completeness if $T = T_{\mathbf{M}}$. Otherwise it is meaningless.

T-linearity: the approach proposed by Höhle and Blanchard provides preservation of [SZP] and [INT] for all left-continuous t-norms. [MAX], however, cannot be satisfied.

S-completeness: in case $T = T_{\mathbf{M}}$, S-completeness coincides with strong completeness (see above). If T does not have zero divisors or if T is a continuous t-norm that is not nilpotent, none of the three fundamental properties is preserved. In case that T induces a strong negation, [SZP] and [INT] are preserved if and only if $N \leq N_T$. If $N = N_T$, S-completeness and T-linearity are equivalent. The mechanisms underlying these findings are always the results for T-linearity. From that point of view, S-completeness does not provide an essential added value compared to T-linearity.

The first important conclusion that can be drawn from these results if we restrict to commonly used t-norms (continuous t-norms and left-continuous t-norms with strong negation): *the three fundamental properties cannot be preserved simultaneously, no matter which t-norm we choose.*

Table 1
An overview of the results achieved in this paper

	strong completeness	S-completeness where (T,S,N) is a de Morgan triple	T-linearity
$T_{\mathbf{M}}$	[SZP], [MAX]	[SZP], [MAX]	[SZP], [INT]
other t-norms without zero divisors (e.g. $T_{\mathbf{P}}$)	*none*	*none*	[SZP], [INT]
t-norms inducing a strong negation (e.g. $T_{\mathbf{L}}$, $T_{\mathbf{nM}}$)	*none*	[SZP], [INT], iff $N \leq N_T$ *	[SZP], [INT]
other continuous t-norms	*none*	*none*	[SZP], [INT]
other left-continuous t-norms	*none*	*???*	[SZP], [INT]

* In the case $N = N_T$, S-completeness and T-linearity are equivalent.

Secondly, as there is no compact axiomatization of maximality in case $T \neq T_{\mathbf{M}}$, the property [MAX] is not achievable anyway. As this is the property that usually has the least practical relevance compared to [SZP] and [INT], T-linearity constitutes a reasonable compromise that preserves these two properties. However, T-linearity is a very weak, non-intuitive, and poorly expressive concept if T does not induce a strong negation. If T does have a strong negation, T-linearity is not just a compromise, but an almost perfect choice, as T-linearity is equivalent to S-completeness for $N = N_T$. This is not just a nice interpretation, it particularly means that even the two independent fuzzifications of the classical linearity concepts (2.1) and (2.2) are equivalent. This result can also be understood as another argument supporting the viewpoint that t-norms inducing strong negations are fundamentally important and beneficial in fuzzy preference modeling [6, 7, 10, 34].

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